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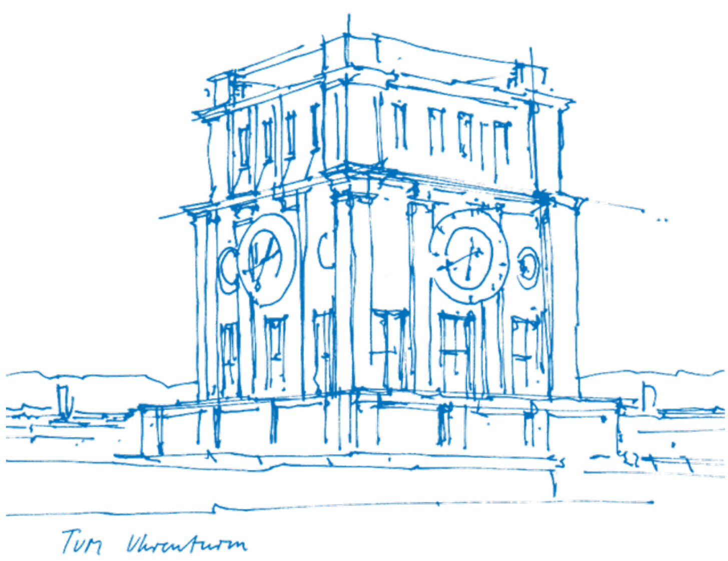
Department of Mathematics
Technical University of Munich

A nonlinear model of Amazon Forest Dieback

Dynamical analysis and overshooting tipping points

Alexander Feil



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Alexander Feil

Thesis for the attainment of the academic degree

Bachelor of Science (B.Sc.)

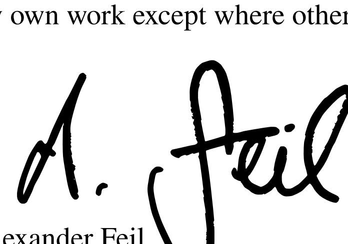
at the Department of Mathematics of the Technical University of Munich.

Examiner:
Prof. Dr. Christian Kühn

Supervisor:
Lucia Layritz

Submitted:
Munich, 21.08.2023

I hereby declare that this thesis is entirely the result of my own work except where otherwise indicated. I have only used the resources given in the list of references.



Munich, 21.08.2023 Alexander Feil

Abstract

The IPCC (2022) predicts that with an increase in average global temperature between 3 and 4°C compared to pre-industrial levels, the Amazon rainforest could experience a tipping point. One of the two main aims of this Bachelor Thesis is to analytically comprehend the existence of this bifurcation and understand its dynamics. For this it analyses a simple nonlinear ODE model of forest dieback, that was originally presented in the paper "Overshooting tipping point thresholds in a changing climate" by Ritchie et al. (2021). Their publication analyses the possibility of "shooting over" said bifurcation thresholds, but still return to a healthy and stable state by sufficiently reducing the global temperature. This thesis does this as well and numerically studies linear overshoot scenarios through a discretisation of the model.
This thesis confirms that Ritchie et al.'s model brings out multiple bifurcations for forest dieback in the Amazon rainforest. Most importantly, there is a saddle-node bifurcation at 8.25°C above the current temperature. Should this bifurcation point be exceeded with a decadal temperature increase of 0.2°C, it would take the forest 19 years to shrink below 10% of its original size. This makes it a *rapid tipping* system. The analysis of overshoot scenarios reveals that returning to a healthy state is theoretically still possible within the model, given the global temperature is reduced ambitiously enough within a given time frame. For example, reducing the temperature by less than 0.01°C per year would not be sufficient if it has been rising by 0.02°C annually for 8 years after the tipping point. Practically, it is still up to question how and to what extent one could reduce global temperature to enable such a turnaround. Therefore, more research in the field of overshoots would help to answer these questions.

Acknowledgements

First and foremost a great thank you to Prof. Christian Kuehn and Lucia Layritz, who have guided me through the writing process of this thesis and have given me the opportunity to work on this fascinating topic. Moreover, I am truly grateful to my companions who have taken their time to help me with writing this thesis - you know who you are and you know how much you helped. Also not to forget is Dr. Florian Lindemann, who gave me the hint to contact Prof Kuehn and follow my passion to combine mathematics with climate studies in this thesis.

Thank you.

Contents

1 Deutsche Zusammenfassung

2019 bezeichnete Emmanuel Macron während der starken Brände im Amazonas den Regenwald als “die Lungen” der Erde, um ihre Bedeutung für den Klimaschutz zu unterstreichen. Mit fortschreitendem Klimawandel wird dieser allerdings zunehmend belastet. Im „Sonderbericht 1,5°C globale Erwärmung” warnte der IPCC (2022)[1], dass der Regenwald bei einer globalen Temperaturerhöhung von 3-4°C einen Kipppunkt erfahren könnte, nach dem ein signifikantes Waldsterben einsetzen würde. In der Theorie Dynamischer Systeme sind solche Änderung der Stabilität als "Bifurkationen" bekannt. Es ist eines der zwei Hauptziele dieser Bachelorarbeit, den eben genannten Kipppunkt mathematisch nachzuvollziehen.

Hierfür wird ein Modell genutzt, das von Ritchie et al. (2021) in der Veröffentlichung "Overshooting tipping point thresholds in a changing climate" (“Kipppunkte übertreten in einem wandelnden Klima”) eingeführt wurde. Es besteht aus einer nichtlinearen gewöhnlichen Differentialgleichung mit einer Variable (Waldfläche v) und einem Parameter (Temperatur T). Es beruht auf Ideen aus der mathematischen Biologie, wie z.B. der Lotka-Volterra Gleichung und TRIFFID[2].

In der Arbeit wird dieses Modell mithilfe der Programmiersprache "Julia" intuitiv erforscht und daraufhin mithilfe der Matlab Erweiterung "MatCont" ein Bifurkationsdiagramm erstellt (siehe Diagramm 5). Dieses offenbart zwei zur Interpretation interessante Bifurkationspunkte: Einen bei globaler Abkühlung um c.a. 13°C und einen bei Erwärhung um 8.25°C. Beide führen über in einen Systemzustand, in dem vollkommenes Waldsterben der anziehende Fixpunkt ist. Sollte sich die globale Temperatur weiterhin um 0.2°C pro Dekade erhöhren, wie es laut IPCC (2022) momentan der Fall ist, wäre letzterer Kipppunkt laut dem Modell in 412.5 Jahren erreicht. Der genaue Bifurkationspunkt liegt in ($v = 65.5\%, T = 34.70$°C) und nach Überschreiten dessen würde der Regenwald ein drastisches Waldsterben erfahren.

Der zweite Hauptteil dieser Arbeit beschäftigt sich mit der Untersuchung der Szenarien nach solch Übertreten der Kipppunkte. Mithilfe eines selbstgeschriebenen, in Ahnang 1 zu findenden Julia Codes wird das Modell diskretisiert und damit das Verhalten des Regenwalds unter verschiedenen Temperaturszenarien numerisch modelliert. Die Auswertungen dessen ergeben, dass bei jährlichem Temperaturanstieg um 0.02°C der Regenwald innerhalb von 18.8 Jahren von dem Bifurkationspunkt auf unter 10% Waldfläche abfallen würde. Dies bestätigt die Ergebnisse von Ritchie et al, welche das System als "schnell kippend" klassifiziert haben. Fortfahrend analysiert diese Bachelorarbeit, inwiefern es möglich ist, ein Absterben bis zu unter 10% trotz Übertretens der Bifurkation zu verhindern. Hierbei werden Szenarien betrachtet, in denen die Temperatur nach Übertritt für für X weitere Jahre um 0.02°C jährlich ansteigt und danach jährlich um Y°C sinkt. Die Codes aus Anhang 1 ergeben, dass ein Abwenden des Absterbens bereits für X>10 Jahre eine sehr ambitionierte Klimapolitik erfordert. Um genau zu sein ist eine *jährliche* Temperaturmilderung um Y = 0.1°C (das fünffache der momentanen Zunahme) nötig. Sollte eine Milderung von jährlich 0.02°C möglich sein, so wäre laut dem Modell eine Reaktionszeit von unter X = 8.35 Jahren ausreichend.

Zusammenfassend kann diese Arbeit Hinweise für die Existenz eines Kipppunktes im Bezug auf das Waldsterben im Amazonas feststellen. Das Übertreten dieser Bifurkation ist bei dem momentanen Temperaturanstieg noch weit in der Ferne, allerdings wäre im Falle eines Übertretens schnelle Reaktion gefordert, um ein vollkommenes Waldsterben zu verhindern, da es sich um ein schnell kippendes System handelt.

[1] Intergovernmental Panel on Climate Change (Zwischenstaatlicher Ausschuss für Klimaänderungen)

[2] "Top-down Representation of Interactive Foliage and Flora Including Dynamics" (Gänzliche Darstellung von Interaktiver Dynamik, die Laub und Flora berücksichtigt.)

2 Introduction

In 2019, Emmanuel Macron called the Amazon rainforest "the lungs" of the earth and thereby stressed it's importance for the global climate (Macron, 2019). With ongoing climate change, the rainforest is coming under increasing stress. The Intergovernmental Panel on Climate Change (IPCC, 2022) warns that "Global warming of 3°C–4°C may also, independent of deforestation, represent a tipping point that results in a significant dieback[3] of the Amazon forest". In mathematics, tipping points are known as *bifurcations* and the following thesis is dedicated to comprehending the existence of said bifurcation point and analysing scenarios where one exceeds this threshold.

This is done using a very simple model that was introduced by Ritchie et al. (2021) in the paper "Overshooting tipping point thresholds in a changing climate". It consists of a nonlinear ordinary differential equation (ODE) with one variable (forest cover v) and one parameter (temperature T). Thereby one can study the effect of changing global temperature on the stability and dynamics of the Amazon rainforest.

This thesis seeks to convey a general understanding for basic methods in climate modelling. Therefore, it first explains core principles behind the design of the model within the methods section. Those mainly come from mathematical biology, including ideas such as the Lotka-Volterra predator-prey models. In the results section it then continues with a detailed bifurcation analysis of the ODE, with an emphasis on both intuition as well as analytical precision. This then results in the presentation of the bifurcation diagram for the entire model. In the discussion section, this is then interpreted and compared to the findings of Ritchie et al. (2021) and the IPCC (2022).

The second main part of this thesis analyses scenarios, in which one exceeds the bifurcation threshold, so-called *overshoots*. Ritchie and his colleagues claim that there is a common flawed assumption that crossing a tipping point would inevitably lead to the the system's collapse (Ritchie et al., 2021). Their paper therefore is devoted to offering counter-arguments to this misconception by demonstrating overshoot scenarios where a collapse is avoided by returning to a stable and healthy state. This thesis presents similar calculations, although with a simpler setting where the temperature varies linearly with time. This is done using a Julia program that uses a discretisation of the ODE to calculate the different paths beginning in the current state of the Amazon rainforest. The scenarios this thesis studies are inspired by the IPCC report on "Global Warming of 1.5°C" and statistics from WWF UK (World Wide Fund for Nature UK). Finally, those results are interpreted and brought into context of previous research on the topic within the discussion section.

It should especially be established that the interpreted results of this thesis are to be appreciated with caution, since the model is indeed very simple and leaves out multiple important factors by making simplifying assumptions - after all, this thesis aims to convey a general understanding of climate modeling and not generate new findings.

3 Materials and Methods

3.1 Literature

This Bachelor Thesis is inspired by and largely based on the paper "Overshooting tipping point thresholds in a changing climate" written by Ritchie et al. (2021). Their paper provides four simple models for ecological systems: The Indian summer monsoon, the ice caps, the AMOC[4] and Amazon forest dieback. Their paper analyses overshoot scenarios for all the above systems. The scenarios studied by Ritchie and his colleagues can vary with multiple factors such as stabilisation temperature, transition time scale and societal efforts in climate action. This thesis uses a simplified linear version instead, that only varies with time and temperature reduction. Further, Ritchie et al. do not provide the calculations that have lead to their results on the bifurcation point and overshoot. The following thesis aims to reproduce these.

Two books that are repeatedly referenced in sections 3 and 4 are "Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering" written by Strogatz (2015) and "Mathematical

[3] The term *forest dieback* shall henceforth be understood as the phenomenon where a forest significantly suffers under new conditions and shrinks.

[4] Atlantic Meridional Overturning Circulation

Biology" by Murray (2001). The former introduces some of the key definitions and results from bifurcation theory that are used in this thesis and the latter presents the concepts from mathematical biology that have influenced the model from Ritchie et al.

3.2 Understanding the model Part 1: The main idea

This section aims to explain the model on Amazon forest dieback introduced by Ritchie et al. (2021) and justify some of it's key elements. In the publication, the model is announced as

"a modified version of the ***TRIFFID*** *model for a single vegetation type[, where] vegetation fraction v is modelled by a* ***Lotka-Volterra*** *equation"*

What is TRIFFID?

TRIFFID stands for "Top-down Representation of Interactive Foliage and Flora Including Dynamics". TRIFFID models are used to predict the behaviour of plant distribution, usually depending on CO_2 fluxes between land and sea (Cox, 2001). The model presented by Ritchie et al. (2021) was designed to study the behaviour of forest cover in the Amazon rainforest with varying local temperature instead of CO_2, making it a modified version of the TRIFFID model. The forest cover is represented by the variable v. It is important to understand that v is a fraction or ratio, such that $v = 0$ stands for 0% cover, i.e. bare soil, while $v = 1$ can be understood as "the full Amazon region being covered in forest".

What is Lotka-Volterra?

The Lotka-Volterra equations (LVE) are a system with 2 differential equations that comes from mathematical biology. Volterra originally designed it in 1926 to model the population of fish species in the Adriatic. In general, it is intended to study the interaction of two species: a predator (represented by population size $P(t)$) and a prey ($N(t)$) (Murray, 2001). The Cambridge Dictionary (2023) defines a predator as "an animal that hunts, kills and eats other animals" and a prey as "an animal that is hunted and killed for food by another animal". The LVE brings the two, i.e. $N(t)$ and $P(t)$, into relation. This is done with the assumption that $\frac{dN}{dt}$ correlates with the natural births of prey (*birth term)* and the predation[5] (*loss term*). For $P(t)$ on the other hand, this predation becomes the *growth term* and the *loss term* is given by a natural death rate. Assuming that these correlations each occur with a constant factor (here: a, b, c and d) yields the following equations:

$$\frac{dN}{dt} = aN - bNP \tag{1}$$

$$\frac{dP}{dt} = cPN - dP. \tag{2}$$

In (1) one can make out the exponential growth term aN . It is exponential, because the equation $\frac{dN}{dt} = aN$ has the solution $N(t) = Ce^{at}$. The predation term bNP correlates with the populations of both P and N, since a larger population of both P(t) or N(t) would intensify the predation and a lower one would make the predation abate. In (2) the same logic applies only with the opposite effect on $\frac{dP}{dt}$.

3.3 Understanding the Model Part 2: The actual model

With this background knowledge it is easier to understand the model proposed by Ritchie et al. Note that in this case, there is only one "species" (the forest) represented by $v(t)$, which is why there will only be one ODE with growth and loss term. The ODE in itself is rather complex, which is why it was split up into three equations that make it more legible. These are now introduced one by one, beginning with the base Lotka-Volterra ODE:

[5] predation is the phenomenon where the predator hunts the prey

$$\frac{dv}{dt} = \underbrace{gv(1-v)}_{\text{growth}} - \underbrace{\gamma v}_{\text{loss}} \tag{M1}$$

Why is the growth term of the form $gv(1-v)$?

A quick explanation would be: saturation. As the forest approaches full forest cover (in mathematical terms $v \to 1$), it will marginally grow slower. Among other things, this could be due to more trees casting more shadow and thereby leaving less space for other plants to grow. Hence, for $v \to 1$, the growth term should shrink. In mathematical biology it is common practice to solve this with *logistic growth*, where the term gv is extended to $gv(1-v)$, since $(1-v)$ becomes dominantly tiny for $v \to 1$, just as needed. The behaviour of the term going to 0 for $v \to 0$ remains.

What is the γ for?

In the explanation of the LVE there was a natural mortality rate d for the predator. γ is exactly that mortality rate for the trees and has been assumed to be 0.2/yr by Ritchie and his colleagues.

What is the g for?

g is the growth factor which is brought into relation with to temperature through the following equation:

$$g = g_0\left[1 - \left(\frac{T_l - T_{opt}}{\beta}\right)^2\right] \tag{M2}$$

Here, T_l is the parameter for varying local temperature, g_0 stands for the fixed maximum growth rate, T_{opt} for the fixed optimal temperature and β is a fixed threshold parameter. Notice how $g = g_0 \iff T_l = T_{opt}$ and $\left[1 - (\frac{T_l - T_{opt}}{\beta})^2\right] \leq 1$, which implies $g \leq g_0 \; \forall \, T_l \in \mathbb{R}$. In other words, the growth rate g is at its peak exactly when the local temperature is optimal for forest growth.
The purpose of β is to give a threshold for T_l beyond which g becomes negative and accelerates the dieback process. This is necessary because beyond a certain difference from T_{opt}, if it's too cold or too warm, the local temperature could have negative impact on the Amazon rainforest. Since g is negative if and only if $g = g_0\left[1 - (\frac{T_l - T_{opt}}{\beta})^2\right] \leq 0 \iff (T_l - T_{opt})^2 \leq \beta^2$, β represents exactly that threshold difference from T_{opt}.

Completing the model

Before introducing the final correlation to this model, it should be established that the Amazon rainforest influences it's own climate. As researchers from the University of California have found, the Amazon triggers its own rainy season through intensive transpiration, which has a cooling effect (Rasmussen, 2017). Hence, the forest cover v has a direct impact on the local temperature T_l. In order to analyse how the rainforest behaves with verying global temperature, which for simplicity shall not vary with the Amazon forest size, one has to make the temperature independent of v.
Ritchie et. al. solve this by defining T as the local temperature under total forest cover[6] (i.e. if $v = 1$, thereby fixing v), and assuming that it linearly declines with tree cover. This yields

$$T_l = T + (1-v)a \tag{M3}$$

which will be referred to as (M3) from now on. Notice how $T_l = T \iff v = 1$, just as it has been defined above, and subbing in $v = 0$ yields $a = T_l - T$. In other words, a stands for the difference between the local temperature for full forest cover and for bare soil. Ritchie et al. take a to be at 5°C. Note that with less vegetation (decreasing v), the temperature rises since $\partial_v T_l = -a$.

[6]Note that Ritchie et al. (2021) refered to the parameter T as "T_f" within their paper.

3.4 The Complete Model

In total, the model can be summarised as:

$$\frac{dv}{dt} = gv(1-v) - \gamma v \tag{M1}$$

$$g = g_0\Big[1 - (\frac{T_l - T_{opt}}{\beta})^2\Big] \tag{M2}$$

$$T_l = T + (1-v)a \tag{M3}$$

Constant	Description	Value
a	Difference in temperature for bare soil and full forest cover	5 °C
β	The threshold to temperatures that are unhealthy for vegetation	10 °C
g_0	Maximum growth rate	2yr^{-1}
γ	Mortality or disturbance rate	$0.2\ \text{yr}^{-1}$
T_{opt}	Optimal temperature for plant growth	28°C

Parameter	Description
v	forest cover. $v \in [0,1]$
T	local temperature for $v = 1$. Parameter to examine. $[T]$ =°C
T_l	local temperature. Varies with T
g	growth factor. Varies with T_l and consequently T.

3.5 Methods for Deriving the Bifurcation Diagram

The derivation of the bifurcation diagram is done in two steps. First, to gain some intuition, this thesis studies two plots that were generated using the coding language *Julia v1.6.3* within the source code editor *VS Code*. The Julia packages used for this were `Plots` (for the plots in general), `LaTeXStrings` (for the axis description with LaTeX writing) and `ColorSchemes` (for the curves' colours. The colour gradient used was "algae"). Any further details within the plots such as arrows or lines were added in "Preview", a pre-installed program on MacOS.
In the second step, the final bifurcation diagram was generated using the Matlab extension *Matcont*. The used Matlab version is 9.12.0.2009381 and for Matcont version 7.1. Within the program, equations *(M1) - (M3)* were first initialised and then the forward and backward paths plotted for a number of starting points within the area of interest. The stability analysis was done with the `ForwardDiff` package in Julia.
All of the methods were carried out on a macOS system.

3.6 Methods to Analyse Overshoot

To generate the paths to study overshoots, the thesis uses a couple of programs that can be found in Annex 1. Those were written in the same version of Julia and VS Code as above. All necessary Julia packages can be found within the Annex.
Within this code, the model is discretised by defining

$$v_{i+1} = F_{T_i}(v_i) := v_i + stepsize \cdot f_{T_i}(v_i)\ \forall i \in \mathbb{N}$$

where f_{T_i} is the right hand side of (M1) after subbing in (M2) and (M3). Notice how $F_{T_i}(v_i) = v_i \iff f_{T_i}(v_i) = 0$. *Stepsize* is the time difference $T_{i+1} - T_i$ between each step in the path. For this it should be said that Ritchie et al. set the model's timescale to be in years, so a $stepsize = 1$ stands for an annual evaluation of the path. Such steps are quite large and could lead to numerical errors, so this thesis sets $stepsize = 10^{-3}$. To analyse specific scenarios, one has to define a vector of temperatures $(T_i)_{i\in 0:N}$, where each entry stands for the value T_i that is needed to evaluate v_{i+1}.

3.7 Key Definitions

Before getting to the dynamical analysis, this section reviews some key definitions from bifurcation theory:

Definition 1 (Fixed Point, Equilibrium State)**.** *p is a fixed point of the system $\frac{dv}{dt} = f_T(v)$, if $\frac{dv}{dt}|_p = f_T(p) = 0$. A synonymous term is "equilibrium state".*

To give an illustrative interpretation of this definition: If the Amazon is in an equilibrium state p, then it stays in this state as time passes, i.e. it will neither grow nor shrink. In mathematical terms, this translates to $\frac{dv}{dt}$ being 0, which is exactly the above definition. Consequently, all states that are not fixed are transient (i.e. the path does not stay there). Paths starting in transient points will tend towards a certain type of fixed points. In the following definition, let $v_0(t)$ stand for the state the path starting in v_0 is in at time t.

Definition 2 (Attracting Fixed Points)**.** *A fixed point p_a is called stable or attracting, if $\exists\ \varepsilon > 0$ such that $\forall v_0 \in B_\varepsilon(p_a)$: $v_0(t) \xrightarrow{t\to\infty} p_a$.*

One can think of the above definition this way: If the Amazon is sufficiently close to a stable fixed state, then after any sufficiently small change in it's size (e.g. deforestation), it would tend towards this stable equilibrium.

Definition 3 (Repelling Fixed Points)**.** *A fixed point p_r is called repelling (synonym: unstable) if $\exists\ \varepsilon > 0$ such that $\forall v_0 \in B_\varepsilon(p_r)$ $\exists t > 0$ for which $v_0(t) \notin B_\varepsilon(p_r)$, i.e. the surrounding states tend away from a fixed point p_r.*

If the Amazon is in an unstable equilibrium state, then a small change in its size would imply a larger diversion from this state.
In nonlinear dynamics, one analyses how the stability of certain equilibrium states behaves for a varying parameter. While in popular science, a change in the stability is called "tipping point", the theory of dynamics refers to them as bifurcations.

Definition 4 (Bifurcations)**.** *A bifurcation is a qualitative change in the dynamics of a system (e.g if the number of fixed points and/or the stability changes). The parameter values where bifurcations occur are bifurcation points (Strogatz, 2015).*

4 Results Part 1: Bifurcation Diagram

4.1 Gaining first intuition

To gain a better understanding of the model's dynamics from ground up, the analysis of the dynamical system commences by solely looking at the differential equation (M1) with varying g. (M2) and (M3) are to be considered later.

$$\frac{dv}{dt} = gv(1 - v) - \gamma v =: f_g(v)$$

Geometric intuition

This section uses a method presented by Strogatz (2015) to gain geometric intuition for the ODE. This is done by drawing arrows that go in the positive or negative direction depending on the sign of the $f_g(v)$-value in this point, which then signify where the path would tend to next. In Figure 1 one can make out two fixed points $p_0 := 0$ and $p_1 \approx 0.9$. The arrows pointing away from p_0 indicate that it is repelling, while the arrows pointing at p_1 show that this is an attracting equilibrium state. Since the degree of f_g is 2, there are no further roots of f_g and hence these are the only equilibrium states.

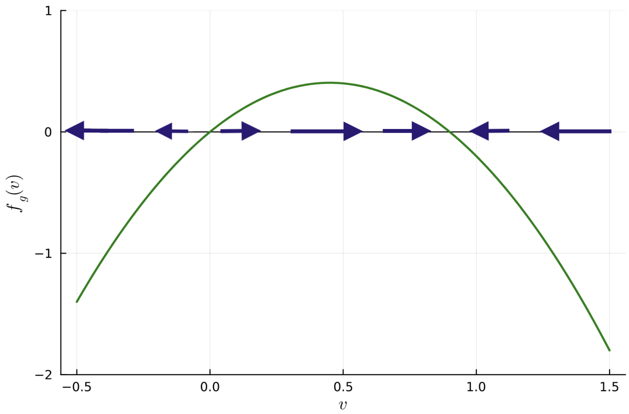


Figure 1: Dynamics of $f_{g=2}(v)$

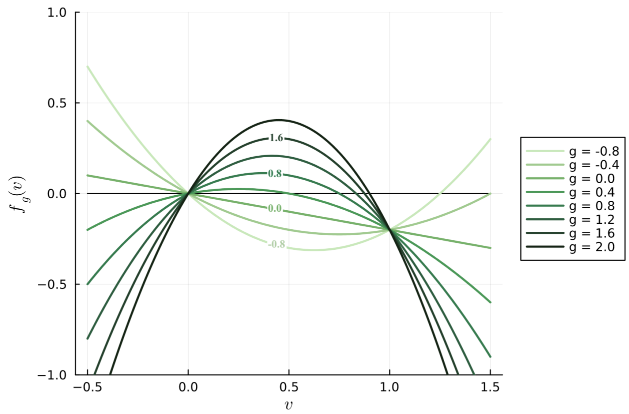


Figure 2: $f_g(v)$ for multiple g-values

In the next Figure 2, the graph of f_g is plotted for a range of values of g. Since (M2) implies that $g \leq g_0$, it only shows those for values $g \leq 2$. Here, there are a few things to notice. First, all plots share the fixed point $p_0 = 0$ as well as $f_g(1) = -\gamma \; \forall g \in \mathbb{R}$. Second, the other fixed point p_1 shifts towards 0 as $g \to 0$, until seemingly $p_0 = p_1$ for $g = 0$. Thirdly, geometric analysis shows that for $g \in \{-0.8 : 0.4 : 2\}$[7], p_0 is repelling and attracting for g ≤ 0 for the other values. Hence, there is a change in stability, i.e. a bifurcation, between 0 and 0.4.

Mathematical Support

This intuition is now to be supported by mathematical arguments. The fixed points can be derived by solving

$$0 = f_g(v) = gv(1 - v) - \gamma v = v\big[g(1 - v) - \gamma\big] = v\big[(g - \gamma) - gv\big]$$
$$\iff v \in \{0, 1 - \tfrac{\gamma}{g}\} =: \{p_0, p_1\}$$

In particular, $p_0 = p_1 \iff g = \gamma = 0.2$. Before checking the stability of these fixed points, recall that:

Lemma 1 (Analytical Implication of Stability)**.** *A fixed point p^* is stable if $\partial_v f_g(p^*) < 0$ and unstable if $\partial_v f_g(p^*) > 0$*

[7] x:y:z stands for the numbers from x to z in steps of y

Intuitive proof of Lemma 1: Let p^* be a fixed point. If $\partial_p f_g(p^*) > 0$, this means that f_g moves through p^* with a positive slope. Since $f_g(p^*) = 0$, this means that $\exists \varepsilon > 0 : 0 < f_g(v) = \frac{dv}{dt}\ \forall v \in (p^*, p^* + \varepsilon)$. Since $f_g(v) = \frac{dv}{dt}$, the states right above p^* have a positive derivative, which means they would tend to a higher value, i.e. away from p^*. In the other direction it works the same way, states right below p^* will decrease next. This translates to p^* being unstable. The opposite logic applies to the the stable state.

Due to the continuity of f_g and the mean value theorem, it follows that between any stable fixed point ($\partial_v f_g(p_s^*) < 0$) and unstable ($\partial_v f_g(p_u^*) > 0$) state, there must be a point p_b^* within (p_s^*, p_u^*) or (p_u^*, p_s^*) where $\partial_v f_g(p_b^*) = 0$. This would be a point where the stability changes, a bifurcation point. This leads to the following Lemma:

Lemma 2 (Analytical Implication of Bifurcations)**.** *Let the stability of the fixed point p^* change in $g*$, i.e. g^* be a bifurcation point for p^*. Then $\partial_v f_{g^*}(p^*) = 0$*

Intuitive proof of Lemma 1: Let the stability of fixed point p^* change in g^*. This means that for a small interval $[g^* - \varepsilon, g^* + \varepsilon]$, p^* is repelling for the values on one half of the ball and attracting on the other. Due to the results of Lemma 1 and the mean value theorem, which one can apply due to $f_g \in C^\infty(\mathbb{R})$, one can deduce that $\partial_v f_{g^*}(p^*) = 0$.

So to confirm the presumptions on stability that were made in the previous section, it remains to calculate

$$\partial_v f_g(v) = (g - \gamma) - gv - gv = (g - \gamma) - 2gv$$

For the fixed point $p_0 = 0$ this implies $\partial_v f_g(v) = g - \gamma = 0 \iff g = \gamma$, which confirms a bifurcation within [0,0.5]. The derivative is negative for $g < \gamma$ and positive for $g > \gamma$, so the fixed point p_0 is stable on the left of the bifurcation and unstable on the right. For p_1 on the other hand $\partial_v f_g(p_1) = (g - \gamma) - 2(g - \gamma) = \gamma - g$, i.e. the exact opposite case. Hence the two fixed points switch stability in the bifurcation, which characterises a transcritical bifurcations. This yields the following bifurcation diagram, which plots the fixed points for each g-value. Dashed branches stand for unstable and solid ones for stable equilibrium states.

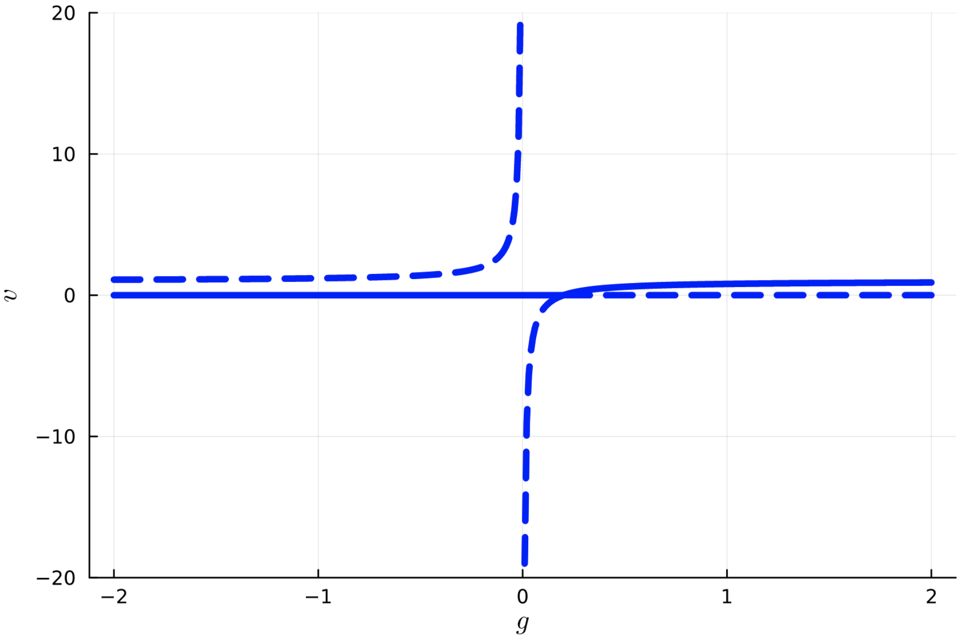


Figure 3: Bifurcation Diagram for (M1)

Notice especially how the two branches switch stability where they intersect - this is where the transcritical bifurcation is. As $g \to 0$ the fixed point p_1 disappears, which matches with the observation that $f_{g=0}(v)$ is linear and has only one intersection.

Temperature comes into play

Now, (M2) will also be taken into consideration, i.e. g is replaced by $g = g_0\left[1 - (\frac{T_l - T_{opt}}{\beta})^2\right]$. The variable v does not appear in this expression, so the dynamics should not change - one can rather think of this as rescaling the horizontal axis. The previous section showed that the bifurcation arose as $g \longrightarrow \gamma$. By introducing (M2), the new parameter to analyse is T_l and its bifurcations can be found by subbing (M2) into the expression of g:

$$g = g_0\left[1 - (\tfrac{T_l - T_{opt}}{\beta})^2\right] \to \gamma \iff (\tfrac{T_l - T_{opt}}{\beta})^2 \to \underbrace{1 - \gamma/g_0}_{>0} \iff |T_l - T_{opt}| \to \sqrt{1-\gamma/g_0} * \beta \approx 9.487.$$

$$\iff T_l \longrightarrow T_{opt} \pm \sqrt{1-\gamma/g_0} * \beta$$

Therefore, one can expect bifurcations for $T_l \longrightarrow (T_{opt} \pm 9.487) = \{18.513, 37.487\}$. Further, inserting (M2) into the expression for the fixed points of f_g that was derived in the previous section, yields the following graph:

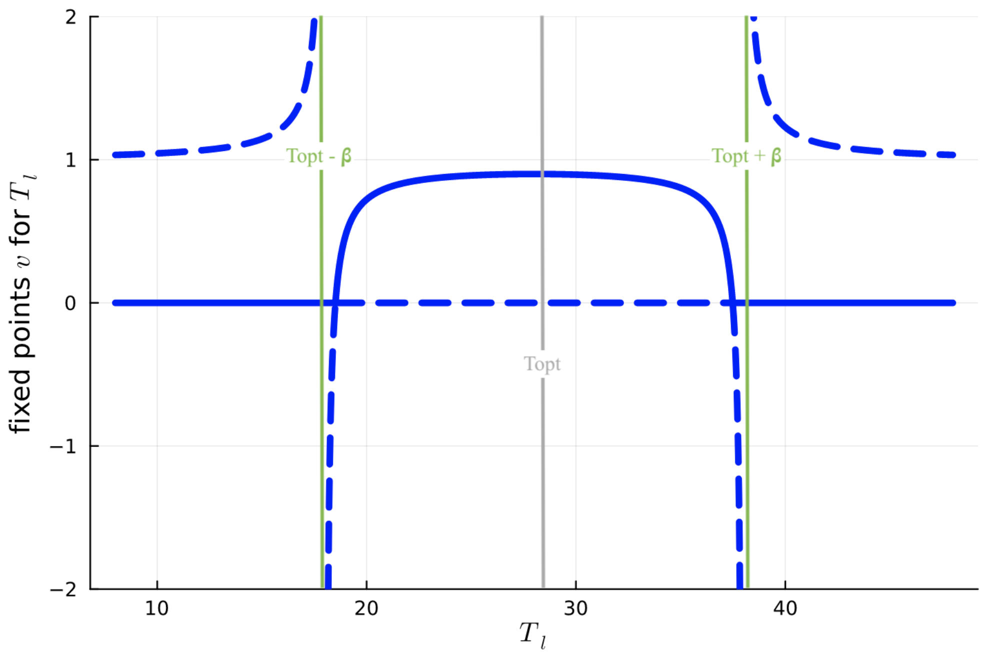


Figure 4: Bifurcation Diagram for (M1), (M2)

This bifurcation diagram has a symmetry around T_{opt}. That is due to the symmetry of the $(T_l - T_{opt})^2$ term and since $g \longrightarrow g_0$ as $T_l \longrightarrow T_{opt}$, the above looks like a version of Figure 3, mirrored around $g = 0.2$. The asymptote at $v = 1$ remained and since $g \longrightarrow 0 \iff |T_l - T_{opt}| \longrightarrow \beta \iff T_l \longrightarrow T_{opt} \pm \beta$ =28°C ± 10°C, there are two vertical asymptotes. As previously explained, this is where g and thereby the growth factor goes negative. Coming to stability, the previous results can be recycled to find that in between the two bifurcations, p_1 is stable and p_0 is unstable. The opposite would be the case outside this interval.

4.2 Bifurcation Diagram for the Complete Model

Now, $(M3)$ is taken into consideration. Inserting (M2) and (M3) into (M1) creates a polynomial of fourth degree that this thesis refers to as f_T. Finding an expression for all the roots of this function in dependence of T would be tedious. Instead, the bifurcation diagram is generated numerically using the program MatCont and subsequently veryfied. For a given starting point, MatCont finds the next fixed point and calculates the branch of fixed points until it reaches a bifurcation. It should be mentioned that the stability in Figure 5 was added manually and its derivation is to be explained later.

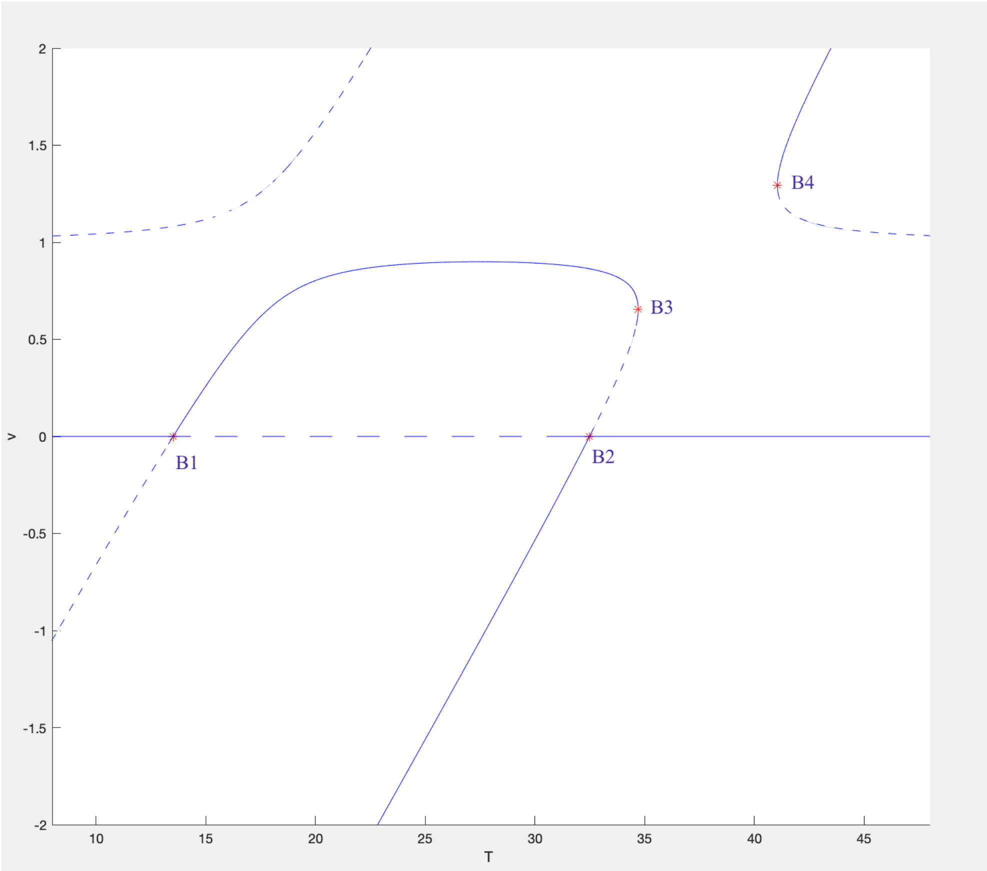


Figure 5: Complete Bifurcation Diagram

There are several things to notice here. First, this plot very resembles a shearing of Figure 4. All curves that existed in the previous figure are still visible here. One can speak of a shearing because horizontal features have remained (e.g. the horizontal asymptote at $v = 1$ and the equilibrium state in p_0) while the vertical asymptotes have been tilted by a certain degree.
In terms of bifurcations, the two transcritical bifurcations at the intersections of the bell with the $v = 0$ line have remained. From here on, they are referred to as $B1 = (0, 13.513167)$ and $B2 = (0, 32.486833)$. Compared to Figure 4, two new saddle-node bifurcations have arisen, $B3 = (0.655472, 34.702013)$ and $B4 = (1.293903, 41.046424)$. Saddle node bifurcations are characterised by bifurcation points T^*, where two fixed points are suddenly generated and only exist for either $T > T^*$ or $T < T^*$, but not both. For the later interpretation it is important to notice that in the interval $[T_{B2}, T_{B3}]$ there are three fixed points $p_1, p_2, p_3 \in [0, 1]$. This actually is the highest number of fixed points that can exist within [0,1] at a time, as the following proposition shows:

Proposition 1. *f_T has at most three roots p_1, p_2, p_3 in the interval [0,1] $\forall\, T \in \mathbb{R}$*

Proof It has previously been established that $f_T(1) = -\gamma \;\forall\, T \in \mathbb{R}$. Further, $f_T(v)$ is a polynomial of fourth degree with positive leading coefficient. This means that there are at most 4 roots and $\lim_{v\to\pm\infty} f(v) = \infty$. Due to the mean value theorem one can deduce the existence of a point $v^* \in (1, \infty)$ such that $f_T(v^*) = 0$. Hence, there can only be at most 3 roots within [0,1].

quod erat demonstrandum.

4.3 Mathematical Verification of Bifurcation Diagram

In this section, the thesis will mathematically verify the completeness of Figure 5 and explain the deduction of the stability as well as the analytical origin of the shearing that makes it differ from Figure 4.

Completeness of the Diagram

To begin with, this thesis will justify that the above plot is in fact complete and that no branches in the bifurcation diagram have been missed.

Case 1: $T < T_{B3} \lor T > T_{B4}$
Using the fact that there are at most four equilibrium states for some fixed value T, as established in Proposition 1, one can conclude that on the left of B3, the bifurcation diagram is complete. That is because for all values $T < T_{B3}$ one can count four fixed points:

(1) $p_0 = 0$, (2) the top left hyperbolic arch and (3+4) the continued lines of the bell curve

Case 2: $T \geq B4$
For $T \geq B4$ the same logic applies, as

(1) p_0, (2+3) the top right hyperbole and (4) the continued top left hyperbole

make the existence of further fixed points impossible.

Case 3: $T_{B3} < T < T_{B4}$
In between B3 and B4, the only visible fixed points are

(1) p_0 and (2) that of the top left hyperbole,

i.e. there could potentially be two more fixed points. If one plots f_T for values $T \in [34, 41]$ one can however see that that those fixed points don't exists. Another option to justify the completeness is to use the fact that Figure 5 is a sheared version of Figure 4, as shown on the next page. Using this knowledge one can reason that there were no further equilibrium states in Figure 4, implying that Figure 5 is complete.

Stability

First, one should establish that the stability on a branch in the Figure 5 will remain the same between two bifurcations. (e.g. on the bell's curve between B1 and B3). That is because if the stability was to change, this would mean that there would be another bifurcation on the branch connecting the two, which would be visible in the above plot. Therefore, one can derive the stability for the three sections on the p_0-line in the above plot by calculating the derivative for (v_0, T_i) with $T_i \in \{10, 20, 35\}$. The package ForwardDiff in Julia yields the following table:

T	11	20	35
$\partial_v f_T$	-1.079	1.62	-1.079

Hence, $p_0 = 0$ is unstable between B1 and B2 and stable outside of this interval. It is interesting to see that the derivatives are the same for $T = 35$ and $T = 11$, since this hints that f_T has a symmetry here.
In fact, knowing the stability of p_0 is enough to complete the entire bifurcation diagram, as the following proposition shows:

Proposition 2. *(Stability of neighbouring fixed points)*
Given that in the system $\frac{dv}{dt} = f_T$

(I) f_T *is continuous*

(II) there are two fixed points $p_1 < p_2 \in \mathbb{R}$ *for some fixed* $T^* \in \mathbb{R}$

(III) for this fixed T^* *there is no other fixed point* $p \in (p_1, p_2)$

(IV) $\partial_v f_{T^*}(v_i) \neq 0$ *for i=1,2*

$\implies$ v_1 *and* v_2 *have opposing stabilities (i.e.* v_1 *stable* $\iff$ v_2 *unstable).*

Proof: Let p_1 be unstable. This means that $\partial_v f_T > 0$. Due to *(I)* there exists some $\varepsilon > 0$ such that $f_T > 0$ $\forall v \in (p_1, p_1 + \varepsilon]$. Intuitively, this means that right after p_1, the values for f_T are above the zero-line. Due to *(I), (III)*, the curve if f_T will remain above the zero-line until p_2. This translates to $f_T > 0 \; \forall v \in (p_1, p_2)$ **(*)**. Looking at p_2, *(IV)* implies that $\exists \varepsilon > 0$: $\frac{dv}{dt} = f_T < 0 \; \forall v \in (p_2, p_2 + \varepsilon)$ **(**)**. Together, **(*)** and **(**)** imply that p_2 is stable. The proof for unstable p_2 is analogous.

quod erat demonstrandum.

This means that knowing the stability for some fixed point (p^*, T^*), one can derive the stability of all fixed points at this T^*-value, given that they are not a bifurcation. In Figure 5, for example, p_0 is unstable between B1 and B2. The next branch above (the bell between T_{B1} and T_{B3} for example) must therefore be stable.

Understanding the shearing

A major difference between Figures 4 and 5 is that a shearing has come into the image. Looking at the difference on the $v = 0$ line, the two plots have their bifurcations here in the points $T_l \in \{18.513, 37.487\}$ and $T \in \{T_{B1}, T_{B2}\} = \{13.513, 32.487\}$, respectively. Those differ by exactly 5, which is a logical consequence of subbing $v = 0$ into (M3):

$$T_l = T + (1 - v)a \tag{M3}$$

since this yields $T_l = T + (1 - 0)a = T + 5$.
In Figure 4, the vertical asymptote was at $T_l = T_{opt} \pm \beta$. Subbed into (M3), this yields

$$T_l = T + (1 - v)a = T_{opt} \pm \beta \iff T = T_{opt} \pm \beta + (v - 1)a$$

This is a linear equation describing the diagonal asymptotes that are visible in the bifurcation diagram. That is because if v changes by one, the T value increases by $a = 5$, which is exactly the case for these asymptotes, as one can confirm by looking at Figure 5. This logic can be applied to any fixed value T_l: On the $v = 1$ horizontal (M3) yields $T_l = T$ - but all other horizontal lines are shifted around the horizontal $v = 1$ by a factor of 5.
In terms of forest dieback, the shearing can be interpreted as the effect of the Amazon on the local temperature that was explained in section 3.3. A larger rainforest will bring down local temperatures, meaning that the thresholds for global temperature T shift with v in comparison to those thresholds for T_l.

5 Results Part 2: Overshoot

This section studies possible overshoot scenarios to understand under what circumstances a tipping event could be avoided despite crossing the bifurcation.

5.1 Finding the Initial State

WWF UK states that the Amazon rainforest covers an area of 6.7 million square kilometres and further mentions that "current figures suggest approximately 17% of the Amazon rainforest has been lost already" (WWF UK, n.d.). From this one could derive that the current forest cover is $1 - 0.17 = 0.83 =: v_0$.
To derive a temperature for the initial state, this thesis uses a dataset provided by NASA (n.d.). This lists the mean monthly temperature between 1970 and 2000 in Manaus, a Brazilian city in the heart of the Amazon rainforest. Throughout the year, the monthly average temperature is at 27°C for 8 out of 12 months and 28°C for the rest. This makes an average annual local temperature of approximately 27.3°C. This would be an initial value for T_l, so to translate to an initial value for T, the term is subbed into (M3):

$$T_l = T + (1 - v)a \implies T_0 = T_l + (v_0 - 1)a = 27.3°\text{C} + (0.83 - 1) \cdot 5°\text{C} = 26.45°\text{C}.$$

Thereby, the initial state is assumed to be at $s_0 := (v_0, T_0) := (0.83, 26.45)$. In the bifurcation diagram, that point is slightly underneath the bell curve (see Figure 10). This is close to a stable equilibrium state, which means that in the model, the Amazon forest cover would tend there.

Scenario 1: Things remain as they are

In 2018, the IPCC published a special report on global warming of 1.5°C. In the summary for policy makers, one can find the following under point A.1.1:

> *"Estimated anthropogenic global warming is currently increasing at 0.2°C (likely between 0.1°C and 0.3°C) per decade due to past and ongoing emissions (high confidence). "* (IPCC, 2022)

For Scenario 1 this means that a constant temperature increase by 0.02°C per year is to be assumed. Note that $T_{B3} - T_0 = 34.7 - 26.45 = 8.25$, which means that the tipping point would be reached after 8.25°C $\cdot(0.02°\text{C/yr})^{-1} = 412.5$ years. To plot the path, one sets

$$(v_0, T_0) := (0.83, 26.45),\ v_{i+1} := F_{T_i}(v_i),\ T_{i+1} = T_i + 0.02 \cdot stepsize \text{ for } T_i < 37$$

The function `plotpath` in Annex 1 then provides the following plot:

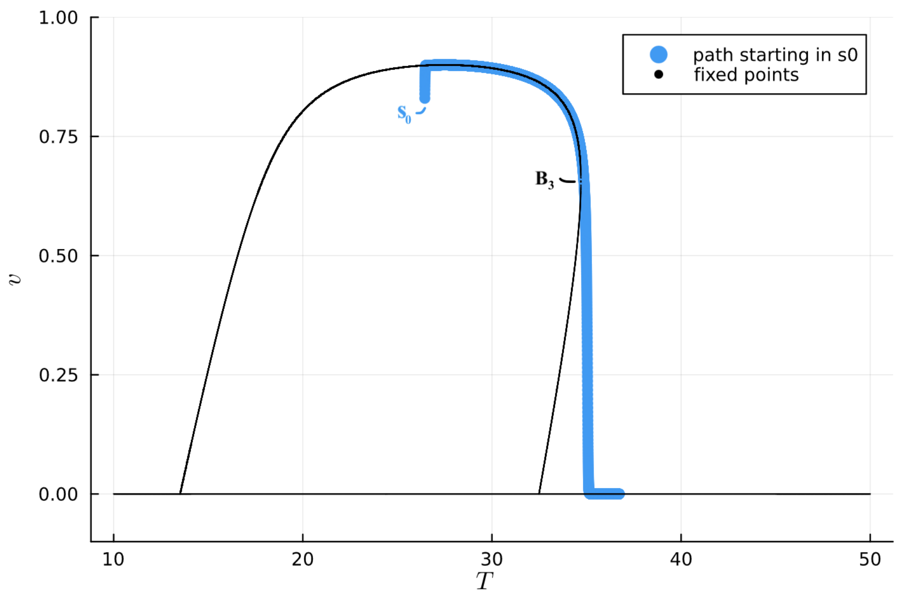


Figure 6: Path for Scenario 1

At first, the path converges to the next attracting equilibrium state[8] and remains on this stable branch until it overshoots the tipping point. It then rapidly falls towards $v = 0$, reaching $v = 0.5$ after another 14 years, falling below 30% of its maximum size 16.9 years after reaching the tipping point and below 0.1 after 18.8 years. Henceforth, a *turnaround* is defined as *unsuccessful* if v falls below 0.1 at any time. Else it shall be called *a successful turnaround*. Further, 18.8 is rounded up to say that the system's *tipping time* is at 19 years.

5.2 Scenario 2: Slight overshoot followed by a linear turnaround

This section analyses under what circumstances a turnaround would be successful. In general, this thesis assumes the temperature to increase by 0.02°C annually for X years. After X *years a turnaround shall occur*, after which the global temperature is linearly reduced by an *annual rate* Y.

Scenario 2.1: Turnaround after half the tipping time

The first attempt analyses whether it would be sufficient to turn around after half the tipping time. This was at 19 years, so X = 0.5 · 19 = 9.5. The temperature reduction rate shall simply be the inverse of the previous increase, so Y = 0.02. In the following plot of this path, the turnaround point is marked in the red box.

[8]Note that the forest remains in a stable equilibrium state for fixed T. When T is varied as it is here, this equilibrium state may vary with it - so this does not contradict the previous intuition of a fixed point.

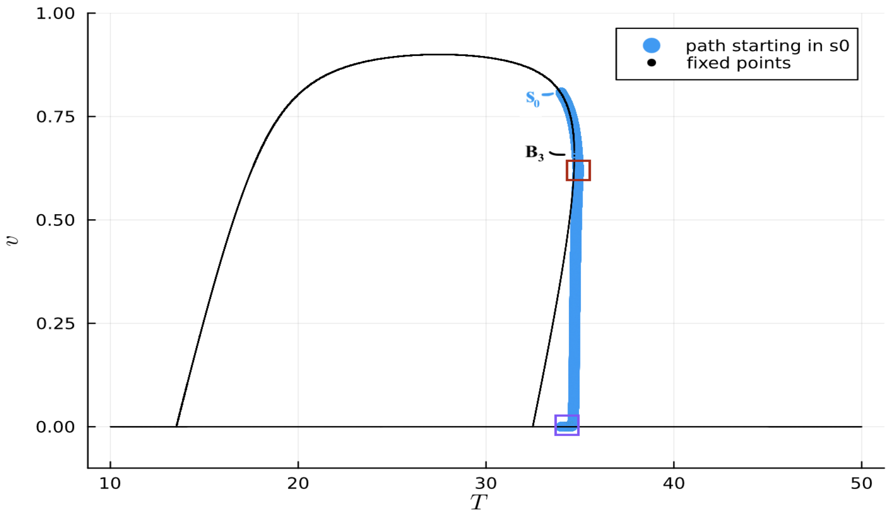


Figure 7: Path for Scenario 2.1

This clearly shows that scenario 2.1 leads to the system's collapse (see the purple box). Hence, a turnaround after half the tipping time and a temperature reduction of 0.2°C per decade is not sufficient. Worth mentioning is that the v-value at the turnaround state is very close to that of the bifurcation point, which shows that in the early stages of the overshoot, the dieback is significantly slower than in the second half. Note that in Figure 6, the path starts in the state that scenario 1 was in when it reached $T = 34$. Since the two scenarios are the same up to this point (the temperature increases by 0.02°C per year in both cases), this practice saves calculation time while not manipulating the result.

Scenario 2.2: Turnaround after less than half tipping time

This opens the question whether there might be a turnaround time X for which a reduction rate of Y = 0.02 is sufficient. Using the code `overshoottest.jl` from Annex 6, one can rapidly check whether a turnaround would be successful. This yields the following table:

X	8	8.25	8.3	8.35	8.4	8.5	9	9.5
successful?	yes	yes	yes	yes	no	no	no	no

So there appears to be a threshold value X^* between 8.3 and 8.35 below which the turnaround is successful. The path for X = 8 and Y = 0.02 looks as follows:

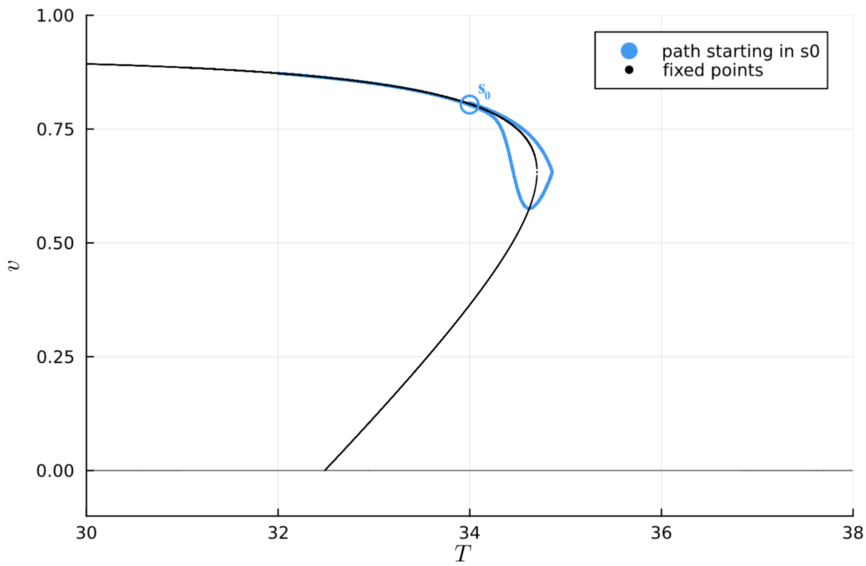
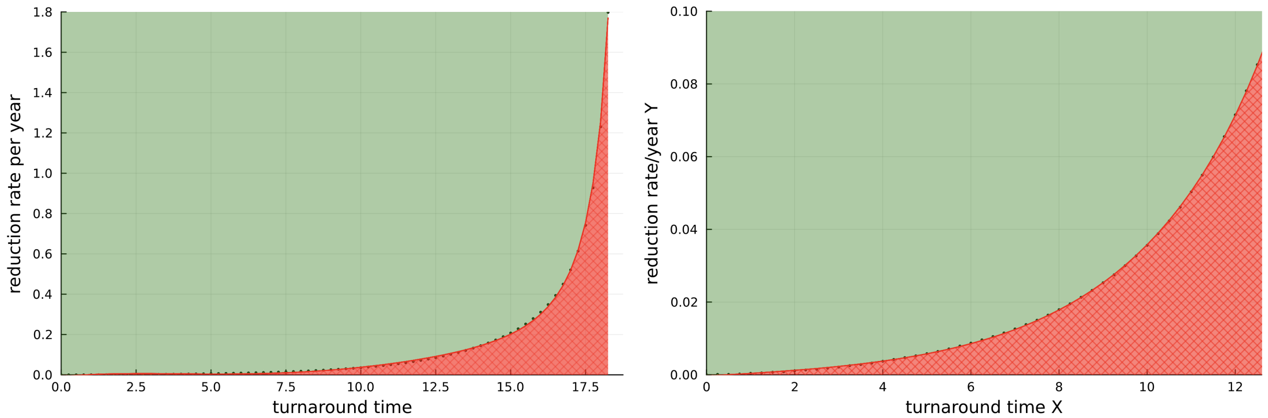


Figure 8: Path for X = 8, Y = 0.02

Turnaround with stronger temperature reduction

As seen in the previous section, there appears to be some threshold value, after which the turnaround becomes successful. In the following section, the code `thresholdplot.jl` in Annex 1 is applied to calculate those threshold reduction rates with a precision of 10^{-3} for X-values between 0 and 18. Since the tipping time was at 19 years, any path with X > 19 would clearly be unsuccessful. The following plot shows the line of threshold points, where all the successful cases above are in green and unsuccessful ones below in red (having a more ambitious reduction rate than necessary implies a successful turnaround)



(a) Threshold Plot on [0,18]

(b) Threshold Plot, zoomed in

Figure 9: Threshold Plots

The curves in the above plots were generated with the method of best fit. They are a linear combination of $\{1, X, X^2, X^3, \frac{1}{(19-X)}\}$, where $\frac{1}{(19-X)}$ has the largest coefficient and hence largest impact. This matches with the observation that a successful turnaround is impossible for X > 19, which hints towards a hyperbolic behaviour. Other fitting attempts with polynomials of degrees up to 10 or an exponential term were less successful.
Notice how the right figure confirms the previous findings regarding the threshold turnaround time X for a reduction rate of 0.02°C per year, which was narrowed down to between 8.35 and 8.4.

6 Discussion Part 1: Bifurcation Diagram

Before getting to the interpretation of the above findings, it should be once again be established that this is a very simple model and therefore the results are to be appreciated with caution. As mentioned in the introduction, there are many more factors coming into play. Therefore, this section aims to convey how the theoretical findings in climate modelling can be applied rather than suggesting actual results for the Amazon rainforest.

6.1 Equilibrium states for fixed temperature

This subsection interprets the bifurcation diagram to understand what developments one can expect the Amazon rainforest to experience for a fixed temperature and initial state $v_i \in [0, 1]$. Within the plot below, the interval for $v \in [0, 1]$ is marked in green and the point (v_0, T_0) derived in section 5 is marked with a dark green dot.

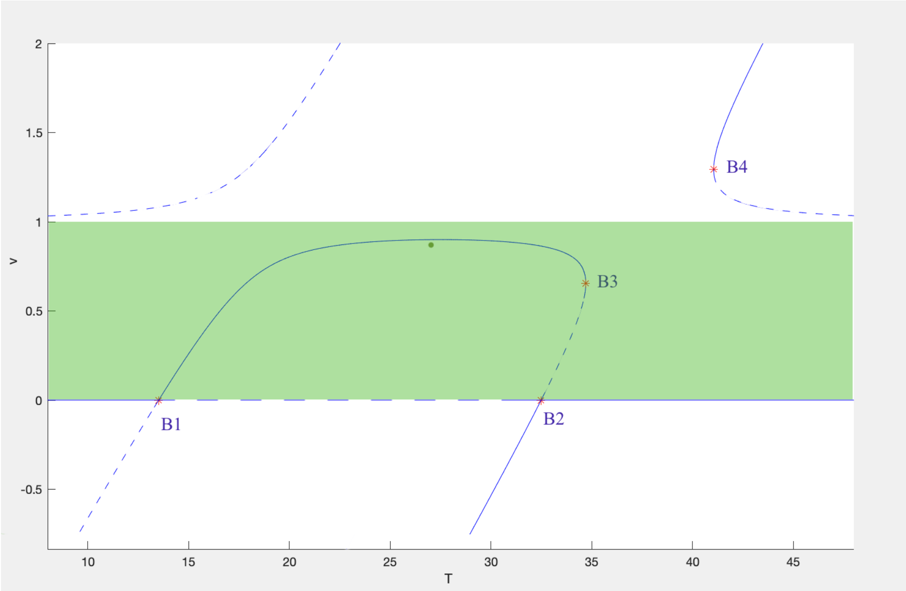


Figure 10: Bifurcation Diagram for Interpretation

The procedure here is the following: Since T is fixed, one can draw an imaginary vertical line above this temperature value and see where this intersects with the branches of the bifurcation diagram. Those would then be the possible fixed points one could converge towards.

Case 1: $T < T_{B1}$
For any initial states $v_i \in [0, 1]$ the two next equilibrium states for fixed T are $p_0 = 0$ and the equilibrium point on the top left hyperbole. Since this hyperbole is unstable and p_0 stable for $T < T_{B1}$, the forest would shrink towards p_0. Hence,the model indicates that temperatures below $T_{B1} = 13.513$ would cause the forest to die back.

Case 2: $T_{B1} < T < T_{B2}$
If the initial value v_i lies within "the bell", then the two neighbouring equilibrium states are p_0 and the one on the bell curve. Since the latter is stable in the interval at question, this is where the rainforest would converge towards. Between $T = 18$ and T_{B2}, the stable branch resembles a plateau around $v \approx 0.85$, so a rather large Amazon forest.
Above the bell, the forest would in fact shrink towards the attracting equilibrium. For T close to T_{B3}, this state would be very little forest cover. The remaining equilibrium states can be disregarded, since they are either unstable or below p_0 and would therefore be "blocked off" by this repelling point.

Case 3: $T_{B2} < T < T_{B3}$
Here there are three possible equilibrium states within [0,1], of which two are stable and one unstable. Therefore, for fixed T, one has to distinguish between v lying above the unstable branch or below. If it does lie above, then the forest would tend towards the stable branch with values for v between 65% and approx. 80%. If it lies below, the forest would shrink towards p_0. Therefore it would be crucial for a successful turnaround to return above the unstable branch connecting B2 and B3 .

Case 4: $T_{B3} < T$
This case is very similar to case 1, since the closest attracting equilibrium state is $v = 0$. The stable equilibrium on the hyperbole above is irrelevant since the unstable branch below "blocks off" any possible growth in this direction. Therefore, also for too high temperatures, the model would predict the forest to die back entirely.

∼

Overall, any initial state v_0 within [0,1] will tend towards an attracting equilibrium state that is also within [0,1]. Below T_{B1} and above T_{B3} that equilibrium state would be the complete forest dieback. Between the two, there is a large interval where the stable equilibrium state is a larger forest cover.

6.2 What could happen for varying temperature

Case 1: The temperature decreases
For values of T between 18 and 30, the equilibrium forest cover is almost constant around $v = 0.85$. However, the curve has a steep slope below this interval, so a small change in temperature would imply a comparably great change in forest size. Already at around $T = 16$, the stable equilibrium state is at only 50%. Assuming that T changes exactly with the global temperature, a global average temperature drop of $\Delta T_{global} = -13.14$°C, so below T_{B1},themodel would predict that the growth of a rainforest is merely impossible.

Case 2: The temperature increases.
In the current reality of climate change, an increasing temperature is more probable. With the bifurcation at $T_{B2} = 32.49$, the model predicts that a tipping point would be reached if the global temperature rises by $\Delta T_{global} = 6.04$°C. Up to this tipping point, the upper stable equilibrium state would be above 65.5% forest cover. If the tipping point was "overshot", according to the model, the Amazon rainforest would tend towards the new attracting equilibrium state of $p_0 = 0$ (i.e. bare soil). In other words, the Amazon forest would start to die back if the global temperature exceeds 32.49°C according to themodel and assumptions. This tipping would happen within a time span of 19 years, which overlaps with the findings of Ritchie et al. (2021). Their paper had found the Amazon to experience "rapid tipping".
The above findings do clash with the findings of the IPCC (2022) that were mentioned in the Introduction, which warned of a tipping point for a temperature increase of 3-4°C. That is significantly lower than the findings. There are multiple reasons for this, one being that the model is very simple compared to the used by the IPCC. Further, it was assumed that a shift in the global temperature would translate one-to-one to the Amazon climate. This assumption is likely flawed, since climate change has different effects on different areas of the world. Ritchie et al. chose to overcome these issues by linking their temperature scale to that of the IPCC findings such that a global temperature change did not translate one-to-one, but rather by a factor such that a temperature increase of 6.04°C in the model lies within the scenario of a 3-4°C increase globally. Such a tipping point would then also be reached quicker with a decadal temperature rise of 0.2°C.

7 Discussion Part 2: Overshoot

The analysis of overshoot has revealed that there indeed is a possibility to overshoot the tipping point but still return to a healthy and stable equilibrium state. Section 5 has shown that after 8.4 years of exceeding the tipping point, it would require a reduction of more than 0.02°C per year for a successful turnaround. In the research to this thesis there were barely any scientific work to find on possibilities to decrease the global temperature (maybe because one currently concentrates on actually stabilising it), so it is difficult to give qualified judgement as to whether this is feasible or not.
As in the previous section, linking this simple model to the findings of the IPCC (2022) with a certain factor would have an impact on the timescale for overshoots. In the same way that the time until the tipping point was reached would decrease, so would the threshold turnaround time X go down, meaning that the actual curve in Figure 7 may be even steeper. Possibilities to compare the results of this thesis to those of Ritchie et al. (2021) are limited, since their paper used more complex overshoot scenarios and different parameteres therein (time over threshold & peak warming overshoot). Despite this, one can make out that both analysis agree on the fact that the options for a successful turnaround become very slim within a decade of overshoot.

Overall, this thesis is able to comprehend the existence of the tipping point for the forest dieback in the Amazon system and confirm the finding of Ritchie et al. (2021) that it is rapid-tipping. However, details such as the distance to the tipping threshold don't overlap with those of the IPCC (2022), which in fact were 'just' "medium confidence". In general, as described by Alizadeh (2022), precise modelling of the climate reality remains a difficult task due to hurdles such as "model uncertainty, which arises due to imperfect numerical representation of different components and processes of the climate system". As seen in section 5 of this thesis, a necessary discretisation of a model can also influence the precision and for much more complex models, one will have to choose between the model's scientific accuracy and numeric run-time. Here, scientific accuracy is meant to stand for the amount of geographic, biological and social processes that were taken into account for the model. The model presented by Ritchie et al. (2021), for example, "only" takes a simplified correlation between the temperature and the forest cover v into account, while also considering that local temperatures vary with v. However, a geographic definition of when a group of trees counts towards this v is missing, as well as a biological distinction between the kinds of trees that are considered here, what different impacts they have and finally how the actions of people living in the region will affect the Amazon. A combination of all these fields is necessary to come closer to a more accurate prediction of climate scenarios and it is interesting to see how this research will develop over the next years.

8 Annex 1

In the following, the codes that were written to numerically derive the paths in section 5 are presented and explained.

modeldefinition.jl

The first code is very simple and has the sole purpose of defining the parameters, $f_T(v)$ as well as the discrete version $F_T(v)$:

```
a = 5 #difference in local temperature for v = 0 and v = 1
b = 10 #threshold where temperature is too high/low
g0 = 2 #maximum growth rate
y = 0.2 #gamma, the mortality rate
Topt = 28 #optimal temperature

f(v,T) = g0*v*(1-v)* ( 1- ((T+(1-v)*a-Topt)/b) ^2 ) - y*v #(M1:3) all together
fdiscrete(v,T) = v + f(v,T) * stepsize #discretisation of f
```

iterations.jl

The second program is supposed to iteratively calculate the sequence of states $s_0, s_1, ..., s_n$. The function `iterations` takes the initial point v_0 and temperature vector T as input, where $\texttt{T}[\texttt{i}+1] = T_i$ (The indices in Julia vectors start with 1). The output is the (length(T) x 2)-matrix `res`, where $\texttt{res}[\texttt{i}+1,:] = [v_i, T_i]$. The stepsize is the one described in the methods section and the variable itself is to be defined in a later program.

```
function iterations(v0, T::Vector) #returns an array res with res[i] is [v at t=i, T at t = i]
    res = [T[1] v0]  # initialise array
    for i = 2:length(T) #for developing T
        res = [res; [T[i] fdiscrete(res[end, 2], T[i-1])]] #calculate v_n+1 = F(vn, Tn)
    end
    return res
end
```

bifurcationdiag.jl

This code generates the bifurcation diagram one can see in the background of the figures on overshoot.
The main idea here is to find the roots of $f_T(v)$ for fixed T (since those exactly are the fixed points) and then plot those using a scatterplot.

```
using Roots, Plots #package needed for findroots
include("modeldefinition.jl") #to load function f

function plotbifurcations(precision = 0.01)
    T = collect(10:precision:40)
    roots = T, [[0.] for i = 1:length(T)] #tuple to store roots

    for k in 1:length(roots[1]) #for all T values, find the roots of f_T(v)
        g(v) = f(v, roots[1][k]) #find_zeros requires g:R->R
        roots[2][k] = find_zeros(g, (0, 1)) #note them in roots[2]
    end

    for k in 1:length(roots[1]) #add the scatterplot to some existing plot
            scatter!([roots[1][k]], roots[2][k]',
                label = false, color = :black, markersize = 1, ma = 0.5)
    end
end
```

plotpath.jl

The following code generates the plot for the desired path. It takes the initial point v_0 and vector T as input and returns the path `x` that was calculated in `iterations.jl` as well as a plot of this path in `pl`. The if-statement in lines 10-12 are there to give the user the option to not have the bifurcation diagram be shown in the background of `pl`.

```
1  include("bifurcationdiag.jl")
2
3  function plotpath(v0, T, background = true)
4      x = iterations(v0, T)
5      plot = scatter(x[:, 1], x[:, 2], markerstrokewidth=0, label = "path starting in s0",
6                  xlabel = L"T_f", ylabel = L"v", ylims = (-0.1,1))
7      if background #if bifurcation diagram wanted in background
8          plotbifurcations()
9      end
10     return x, plot
11 end
```

"overshoottest.jl"

This code was written in order to analyse the success of a turnaround scenario in a time-saving manner. While the function `iterations` memorised the entire path, this is not necessary here. As described in section five, this code takes the turnaround time X and reduction rate Y as input and return an array listing the result of the turnaround. Note that the iterations starts in `v34` to save calculation time, as described in scenario 2.1.

```
function overshoottest(X, Y)
    Tover = [collect(34:0.02*stepsize:34.702+X*0.02); #up to turnaround with 0.02
            collect(34.702+X*0.02:-Y*stepsize:33)] # after turnaround with rate Y
    res = [Tover[1] v34] #starting in T = 34 to save calulation time
    for i = 2:length(Tover)
        res = [Tover[i] fdiscrete(res[2], Tover[i-1])] #calculate v_n+1 = f(vn, Tn)
        res[2] > 0.1 ? continue : break #break if below 0.1
    end
    res[2] > 0.1 ? test = true : test = false #return true if return successful
    return ("overshoot time $X", "reduction rate $Y", "successful? $test")
end
```

userinterface.jl

All the above codes are combines in this file for a better overview. Here one can load all the functions that were previously explained and the `stepsize` as well as initial points can be defined here.

```
using Plots, LaTeXStrings
include("modeldefinition.jl")
include("iterations.jl")
include("plotpath.jl")
include("overshoottest.jl")

stepsize = 1e-3 #stepsize = 1 if fdiscrete is evaluated every year
(v0, T0) = 0.83, 26.45 #initialising the current state
(v34, T34) = 0.808, 34
```

thresholdplot.jl

This final code was written to find the threshold values described in section 5.1. It is ot included in the above `userinterface.jl` because there is to much one could potentially change to summarise it as a function. What happens here is that for a selection of X-values, the threshold Y* is searched by starting in 1e-5 and increasing that value in the while loop until the turnaround is successful. This then is saved as the threshold value.
The resulting numbers are then plotted and a best fit can be generated using the packages GLM and DataFrames.

```julia
using GLM, Plots, DataFrames
include("modeldefinition.jl"); include("overshoottest.jl")
stepsize = 1e-3; (v34, T34) = 0.808, 34

interval = 0.25:0.25:18.25 #interval for values of X
threshold = [0 1e-5;collect(interval) zeros(length(interval)).+1e-5]
    #^ matrix with X values starting in 0 on left and threshold values on right

for k in 2:size(threshold,1) #for X values
    threshold[k,2] = threshold[k-1,2] #start at previous value to save time
    while !overshoottest(threshold[k,1], threshold[k,2])[3] && threshold[k,2] < 2 #while unsuccessful & smaller than 2
        threshold[k,2] += 0.0001 #try again for higher value
    end
end

p = scatter(threshold[:, 1], threshold[:,2], xlims = (0, 18.25), ylims = (0, 1.8),
            yticks = (0:0.2:1.8), xticks = (0:2:18.25),
            color=:green, legend = false,  ms = 1,
            xlabel = "turnaround time X", ylabel = "reduction rate/year Y") #plot Y*

df = DataFrame(x = threshold[:,1], y = threshold[:,2]) #dataframe for linear model
model = lm(@formula(y ~ 1+x+x^2+x^3+1/(19-x)), df) #best fit
p = plot!(df.x, predict(model, df), fillrange = [1.8], fillalpha = 0.4, color=:green) # mark successful area green
p = plot!(df.x, predict(model, df), fillrange = [0], fillalpha = 0.4, color=:red) # unsuccessful red
p = plot!(df.x, predict(model, df), fillrange = [0], fillalpha = 0.4, color=:red, fillstyle = :x) #line pattern in red are
```

9 Glossary

Vocabulary

Bifurcation	Where a qualitative change in the dynamic of a system occurs.
Bifurcation Point	The value of the parameter where the bifurcation occurs.
Equilibrium state	A synonym for bifurcation point.
Forest Dieback	the phenomenon where a forest significantly suffers under new conditions and shrinks.
Overshoot	A scenario where one exceeds the bifurcation point of interest.
Stable Fixed Point	An equilibrium state that attracts paths that start sufficiently close.
Stepsize	The time difference for a step within the discretisation of the path generation.
Tipping Point	A synonym for bifurcation point.
Turnaround	The attempt to reduce the temperature after overshooting the tipping point in order to return to a healthy and stable state. This thesis regards it as unsuccessful if the forest shrinks below 10% of its maximum size, else it is successful.

Abbreviations

ODE	Ordinary Differential Equation
LVE	Lotka Volterra Equations
x:y:z	the numbers from x to z in steps of y.

Notation

g	Varying growth rate in (M1)
g_0	Maximum growth rate
T	Local temperature for v = 1
T_0	Initial temperature, derived in section 5.1
T_{Bi}	Bifurcation points within the model (See Figure 5)
T_l	Local temperature
v	Forest Cover
v_0	Initial forest size, derived in section 5.1
X	The time after the threshold where a turnaround commences
Y	Annual temperature reduction rate after after the turnaround

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