Mathematical Analysis of a Board Game

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Contents

Abstract

Have you ever played a board game and for the longest time, you did not find a winner? In August 2024, during my volunteering in Kenya, I came across a board game that apparently took 'too long to win'. To understand why this is the case, this thesis borrows tools from the Markov Chain Theory to find that the expected playing time lies at approximately 42 turns per person. Further, it analyses the probabilities of winning /losing the game, concluding that it is almost twice as probable to win rather than land on the 'global extinction' field.

A Word of Thanks

A great thank you to my friend Marc Belorgey, who has been supporting me during this research and gave me the motivation in times of hardship to continue with this research. Most essential to this paper however is the lady who introduced me to the game and told me the story of the unfinished game that inspired the initial idea of conducting the research.

1 Motivation

In August 2024, it was brought to me that a practice round of the newly acquired board game had taken so long that no one finished and it was given up. This sparked an interest to investigate the fairness of the game for the sake of future students who might get frustrated by the game - which would defeat the purpose.

2 Introduction

For the sake of not offending the creators of the boardgame, its name will not be mentioned. What really matters is that it is a themed version of snakes-and-ladders. The additional features of the game do not change the movement of a player compared to the original snakes and ladders, with only one exception: A "global extinction" field was added, which directly eliminates a player from the game. This thesis will mainly answer two questions:

1. How likely is it to win the game vs land on the global extinction?
2. What is the expected number of turns to finish the game?

To accomplish this, results of Markov Chain Theory will be applied.

One may say that using Markov Chains for this purpose seems like taking it a bit far. I would say that I absolutely agree and can only add that it was a great joy to finally apply the theory I had learnt at uni in the real world.

3 Methodology

This section will present the rules of the game, the mathematical background in Markov Chains used and the calculations that lead to the results of this paper.

3.1 Rules of the game

The goal of the snakes and ladders game is to move ones character by the throw of a six-sided dice through a board of 100 fields and make it to position 100. After the first throw of a dice, the player starts on fields 1 through 6 with probability $\frac{1}{6}$. On every following move, the dice is thrown and the character advances by the number shown.
Amongst the 100 fields, there are so-called "snakes" and "ladders". The snakes put a player back to a lower field, while ladders advance a player to a higher field.
Field number 81 is the global extinction field - meaning that the player is directly excluded from the game.
Field 100 is the winning field - the player has successfully completed the game.
Should a player be on fields 95-99 and throw a number that would place them higher than the field 100 (e.g. getting a 4 on field 98), the player stays on the current field and their turn is over.

3.2 Basics of Markov Chains

The Markov Chain Theory provides tools to analyse probability games like the snakes and ladder games. In simple terms, the Markov condition says that the probability of the next event must only depend on the current state and not any previous events. This is the case for the conservation game: To know where a player will advance to only depends on the field they are currently on - it does not matter how they get there.
Hence, it is possible to construct a matrix for the markov chain that models the game. This matrix will be referred to as M and stores the probabilities to move from one particular field to another. For example, moving from field 2 to field 3 happens with probability $\frac{1}{6}$. Translated into markov chain vocabulary, this means that M[2,3] $= \frac{1}{6}$, i.e. the entry in row 2 and column 3 is $\frac{1}{6}$.
This matrix has a dimension of 100 x 100, meaning it has 100 rows and 100 columns, one for each field in the game. Let the initial distribution of the game be $\nu =$(1/6, 1/6, 1/6, 1/6, 1/6, 1/6, 0, ..., 0), i.e. one starts on fields 1 through 6 with probability 1/6. Then, to find the probability to land on a specific field after n moves, one has to calculate

$$\nu' \cdot M^{n-1}$$

The k-th entry of the vector resulting from the above calculation gives the probability of landing on field k after n throws of a dice.

3.3 Implementation

This paper has carried through the calculations using the programming language julia.
With the first step, the program written for this thesis defines the matrix m as well as the initial distribution:

```
using Plots

m = zeros(100,100)
v = [1/6 * ones(6); zeros(94)]
```

In the second step, the snakes and ladders are not yet regarded. This leaves the matrix with the simple structure that for a field n between 1 and 94, the probability to go to the next 6 fields is 1/6. This is implemented as follows:

```
for i = 1:94
    for dice = 1:6
        m[i,i+dice] = 1/6
    end
end
```

The fields 95 to 100 are purposefully excluded, since they are subject to the rule that going over 100 is impossible. For field 98, for example, the probabilities change to $\mathrm{M}[98,99] = \frac{1}{6}$, $\mathrm{M}[98,100] = \frac{1}{6}$ and $\mathrm{M}[98,98] = \frac{4}{6}$. Further, the player is finished on fields 81 or 100 and would not move away from them. These exceptions are considered in the following code:

```
#field 95
m[95,95] = 1/6
m[95,96] = 1/6
m[95,40] = 1/6
m[95,98] = 1/6
m[95,6] = 1/6
m[95,100] = 1/6

#field 96
m[96,96] = 2/6
m[96,40] = 1/6
m[96,98] = 1/6
m[96,6] = 1/6
m[96,100] = 1/6

#field 97 stays empty

#field 98
m[98,98] = 4/6
m[98,6] = 1/6
m[98,100] = 1/6
#field 99 empty

#field 100
m[100,100] = 1

#global extiction
m[81, :] = zeros(100)
m[81,81] = 1
```

The next and final step for the matrix alteration will add the snakes and ladders to the matrix. To accomplish this, all snakes and ladders are saved as tuples, which first list the field that transports a player and then where a player is transported to. Then, the code goes through the matrix, and for every possible step that leads to a ladder or snake, the entry is changed to make the player directly move down or up the ladder.

```
snakes = [[37, 2], [57, 26], [70, 9], [76, 21], [88, 46], [94, 64], [97, 40], [99, 6]]
ladders = [[7,42], [11,45], [15,72], [19,51], [23,55], [28, 58], [30,59], [34, 61], [38, 65], [50, 75], [66,85], [74,89], [77,93], [82,100]]

for snake in snakes
    m[snake[1],:] = zeros(size(m,1))
    for i = 1:100
        if m[i,snake[1]] > 0 #if it is possible to go to this field
            m[i,snake[1]] = 0
            m[i,snake[2]] = 1/6 #alter the probability to go to the terminus of the snake
        end
    end
end

for ladder in ladders
    m[ladder[1],:] = zeros(size(m,1))
    for i = 1:100
        if m[i,ladder[1]] > 0 #if it is possible to go to this field
```

```
            m[i,ladder[1]] = 0
            m[i,ladder[2]] = 1/6 #alter the probability to go to the terminus of the ladder
        end
    end
end
```

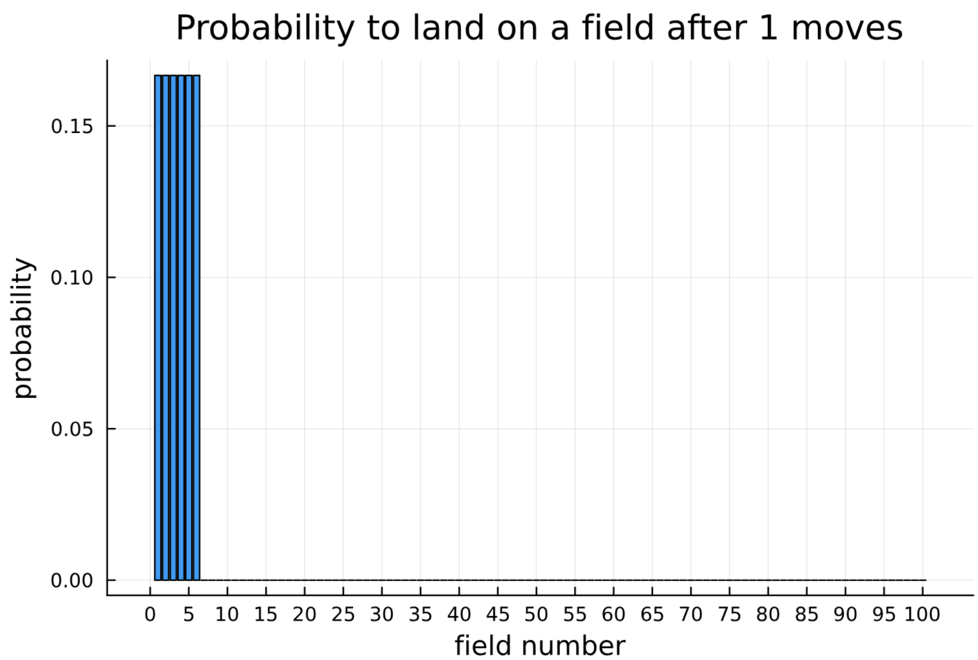
To produce a bar plot that shows the probabilities of landing on a specific field after n moves, the following function is used:

```
function plot_field_probs(moves)
    probabilities = v'*m^(moves-1)
    bar(1:100, probabilities', ylabel = "probability", xlabel = "field number", legend =
    false)
    xticks!(0:5:100)
    title!("Probability to land on a field after $moves moves")
end
```

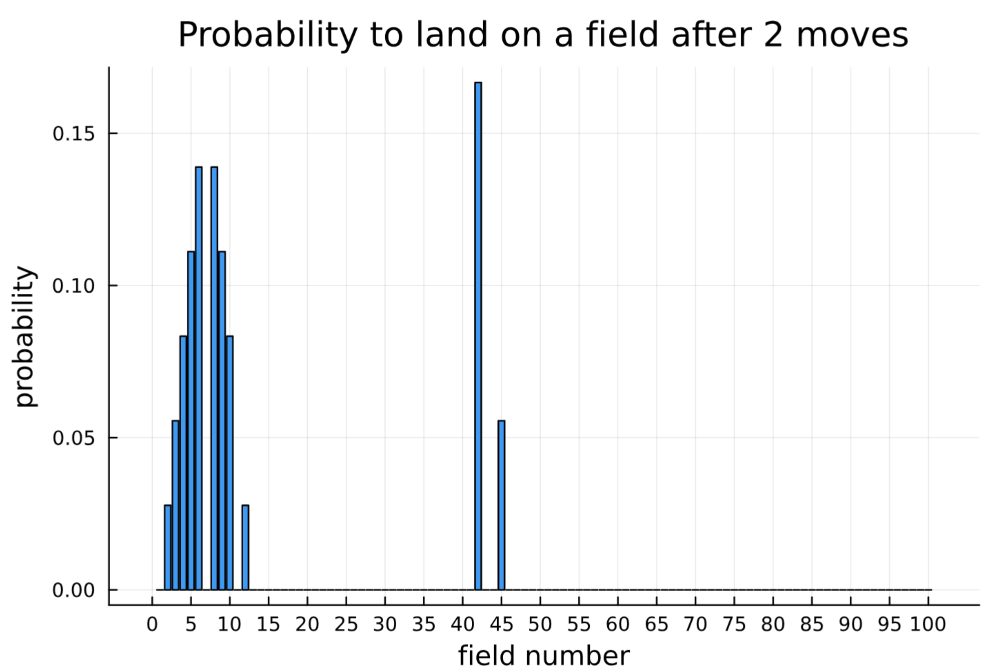
4 Results

The following section will show the results of this paper:

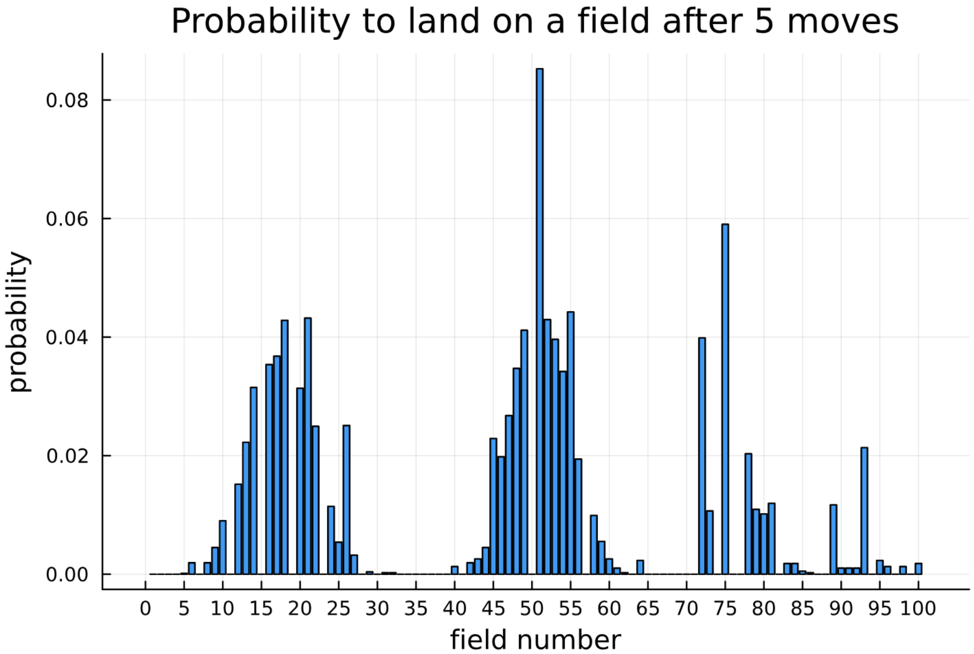
After the first move, the probabilities are quite clear: A player will land on fields one through six with a probability of $\frac{1}{6}$. This is shown in the following plot:



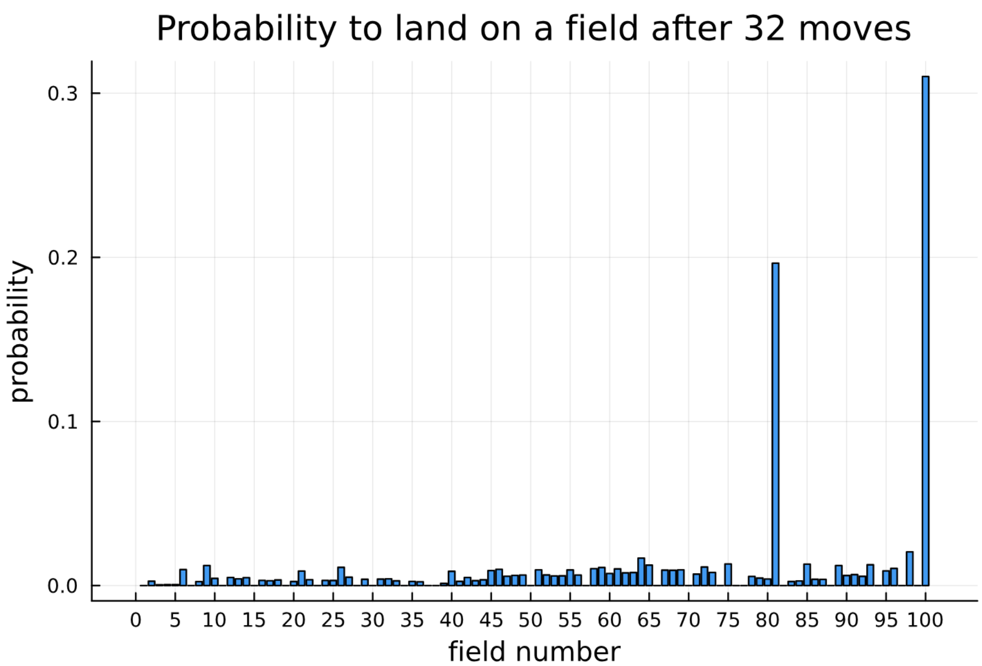
After two moves, with the help of two ladders, one can already get quite far:



The first possible time to reach field 100 is after 5 moves, at a probability of 0.18% (see below). For the global extinction the first possiblity comes after 4 moves at a probability of 0.15%, whereas after 5 moves it is already at 1.2%:



After 31 moves, the probabilities to finish on field 100 are 30% and 19.1% for global extinction on field 81. After 32 moves, those probabilities are at 31% and 19.6%, respectively, which means that after 32 moves one has a chance above 50% to finish the game.



Since calculating $lim_{n \to \infty} \nu' M^n$ is quite pointless for application, this paper will limit itself to the probablilities of finishing on a specific field after 1000 moves. For all fields except 81 and 100, the

likelihood to be there lies below 0.0001%, i.e. neglegibly low. Of possible games within 1000 moves, 36.2% will end in global extinction and 63.8% of the time will the player successfully reach the winning square.
However, even playing 1000 moves is unlikely, which is why the probabilities for certain cornerstones are presented in the table below:

#moves	50	75	100	150	200	1000
P(square 81)	26.4%	31.3 %	33.7%	35.6%	36.0%	36.2%
P(square 100)	44.4%	54.1 %	59.0%	62.6%	63.5%	63.8%

4.1 Expected Playing Time

The formula to calculate the expected number of turns is

$$E[T] = \sum_{n\in\mathbb{N}} n \cdot P(T = n)$$

In the following calculation, let N be the number of moves needed to finish the game, F_n be the field reached in move n and $\nu' * M^n[81]$ be the 81st entry of the vector $\nu' * M^n$. As an approximation, only cases of $n \leq 1000$ will be considered.

$$\begin{aligned}
E[N] &= \lim_{k\to\infty} \sum_{n=0}^{k} n \cdot P(N = n) && (1)\\
&\approx \sum_{n=1}^{1000} n \cdot P(N = n) && (2)\\
&= \sum_{n=1}^{1000} n \cdot P(\text{squares 81 or 100 are reached for the first time in move n}) && (3)\\
&= \sum_{n=1}^{1000} n \cdot P(\text{squares 81 or 100 are reached in move n and have not been reached before}) && (4)\\
&= \sum_{n=1}^{1000} n \cdot P(\text{squares 81 or 100 are reached in move n and are not reached in move n-1}) && (5)\\
&= \sum_{n=1}^{1000} n \cdot \left[P(F_n \in \{81, 100\}) - P(F_{n-1} \in \{81, 100\}\right] && (6)\\
&= \sum_{n=1}^{1000} n \cdot \left[(\nu' * M^n[81] + \nu' * M^n[100]) - (\nu' * M^{n-1}[81] + \nu' * M^{n-1}[100])\right] && (7)\\
&\approx 41.89 && (8)
\end{aligned}$$

(1) is the definition of the expected value
(3) is the same as saying that the game ends after n moves - since if the points had been reached before, the game would have ended beforehand.
(5) is possible because once one has reached field 81 or 100, one will stay there. Hence, if they have been reached in a step before n-1, one will also be there in step n-1.
(6) is possible because the (5) describes the set of all cases where the game is finished after n moves, excluding those that had already finished after n-1 moves.

To calculate the expected playing time, this paper assumes that the throw of a dice and playing takes approximately 5s (throwing the dice) + 5s (moving the character) + 2s (passing on the dice) + 8s (explaining the game's theme details) = 20 seconds. Multiplying the expected number of moves 41.89 with the expected time per moves gives an expected game duration of 41.89*20s ≈s 837.8s = 13min 58s for 12 seconds per move.

5 Conclusion

The purpose of this paper was to see whether the design of the snakes and the ladders, in addition with the global extinction field, were leading to a game that was too frustrating to play.
Firstly, it can be concluded that in the long run it is almost twice as likely to win the game than to lose to global extinction. Whether this is too much or too little will be left to interpretation by the reader.
Secondly, this paper finds that the expected turns to finish the game is at 41.89. Assuming an average time of 20 seconds per move, this would add up to almost $\frac{1}{3}\text{min/turn} * 41.82\text{turns} \approx 14$ minutes per player. Should one play the game in a group of 5 students, which indeed is possible, this would add up to more than one hour.

6 Outlook & Further Research

This thesis calculates the probability for a player to win/lose a game as well as the expected number of turns to finish the game. However, the expected duration of the game in minutes could be computed with more precision. For this, research of the student behaviour between turns needs to be conducted. Out of personal experience, I would expect the duration of each throw of a dice to decline with an advancing game.
Further, should it be decided to alter the game in order to decrease the expected playing time, it would be necessary to model different alterations and find the version that leads to the greatest pleasure for the player.