graph theory

Alexander Feil Scholarship

Chapter 1: The Euler Walk

Leonhard Euler

Graph Theory was invented in the 18th century by German mathematician Leonhard Euler. He started to think about it when he was given the "seven bridges problem" by the king of Königsberg. The king wanted to know, if he could start in his castle, cross all bridges exactly once and get back to his castle. To make it easier, Euler started to make little images to represent the bridges and islands with just lines and dots and thereby started to solve the first problem of graph theory.

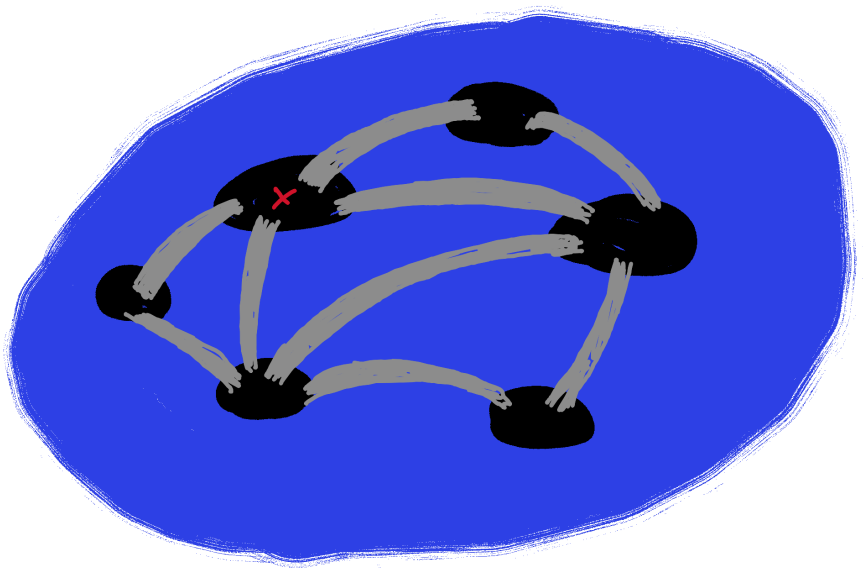
Nowadays, graph theory is used in GPS systems to find the fastest way to your destination, companies use it to find the cheapest way of getting their items shipped and AI is based on ideas from graph theory.

Problem 1: The Euler Walk

In the picture below, you see a few islands with bridges in between them. What you have to do is find a Euler Walk starting on the island with the X. This means that

1. You have to start on the island with the X
2. You have to cross **every** bridge exactly **once**
3. You are allowed to visit one island multiple times
4. You must finish on the island with the X.

Good luck!



Solution 1: The Euler Walk

Did you find a solution you think is right? Let's see how Euler would have done it.

Step 1: Turn the image into a **graph** with dots for the islands and lines for the bridges

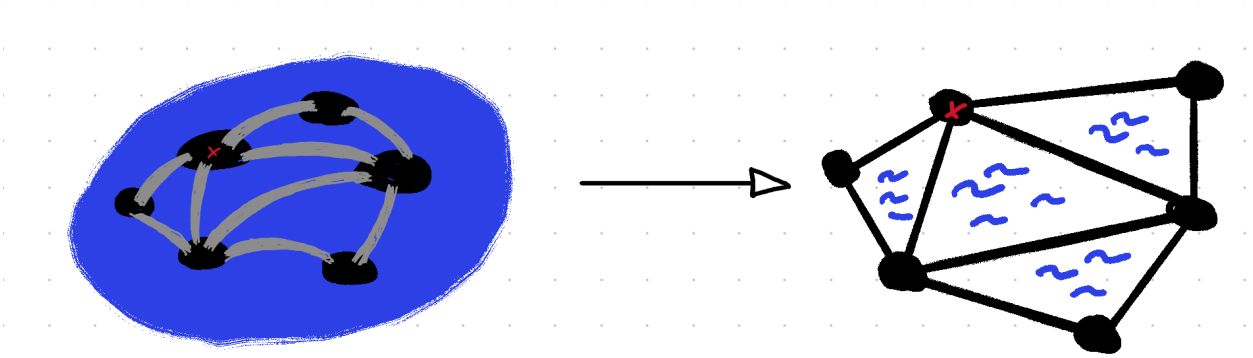
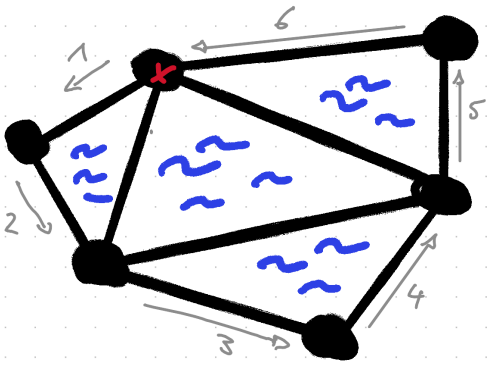


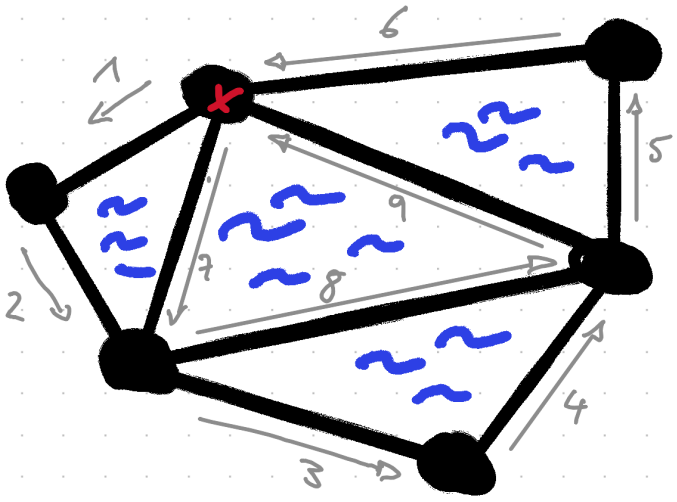
Image Graph

Step 2: Find a Euler Walk in the graph

Then you could continue by starting your path around the outside:

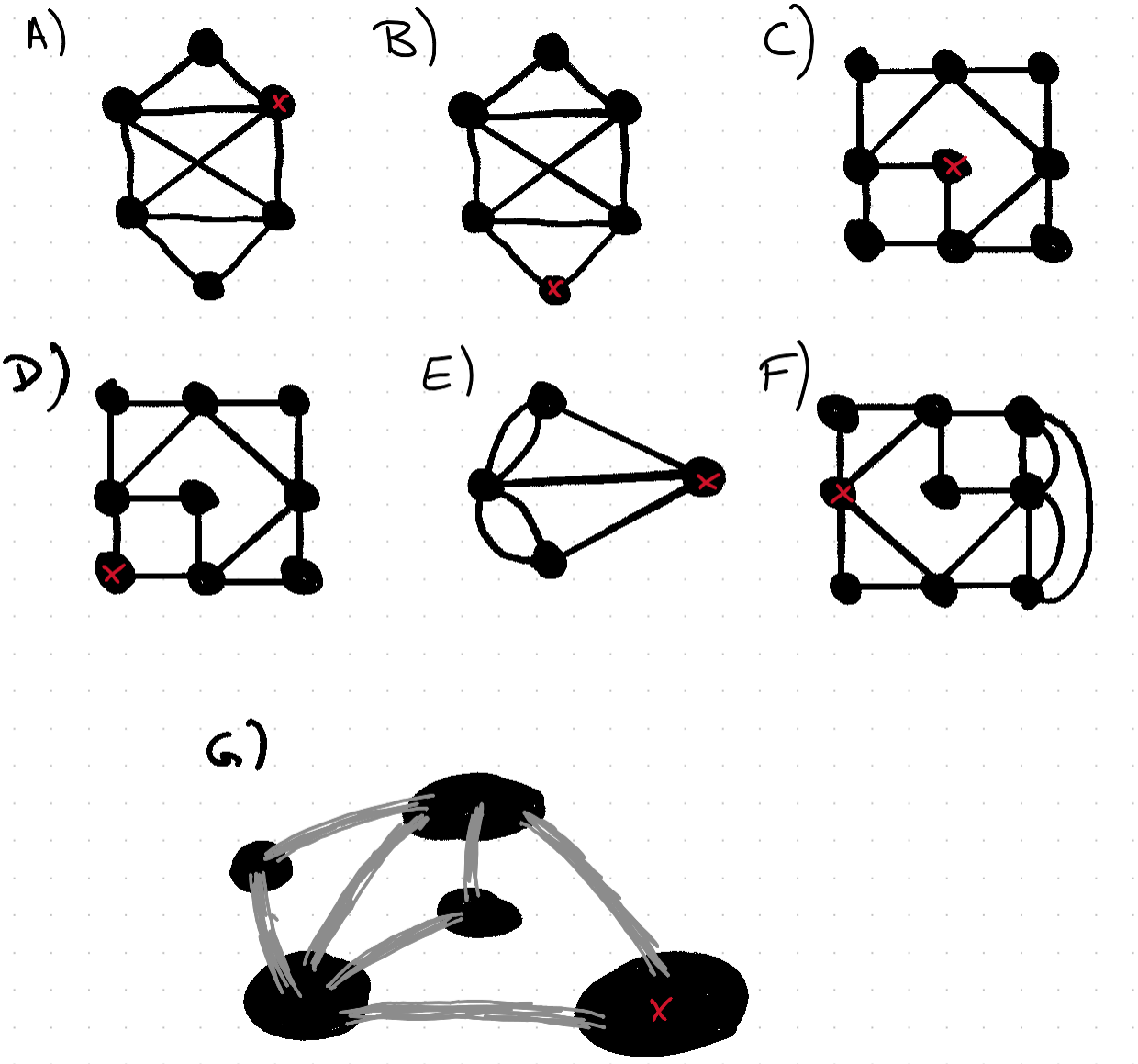


From here, the rest is easy:



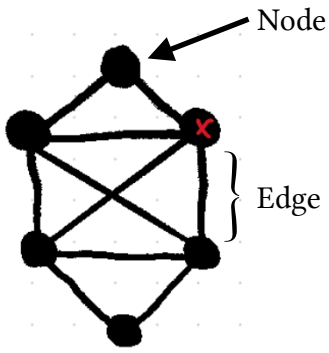
Exercises Chapter 1

Find the Euler Walks for the following graphs, starting on the X. Watch out!!! One of them is impossible.



Chapter 2: What is a Graph?

Graphs are a mathematical concept that we can use to make models of real life situations. They consist of points (we call them **nodes**) and lines that connect the nodes (we call them **edges**).



These little graphs are very simple and that makes them so useful. For example, look at the two images below of the public transport system in the city of Vienna, Austria. Which one is easier to read?

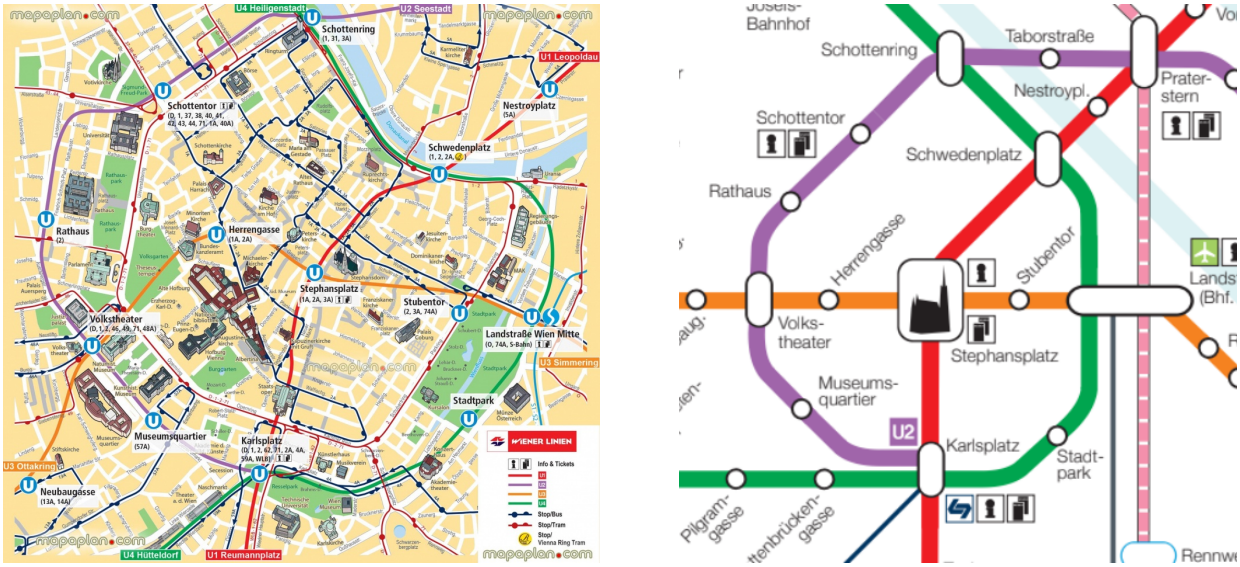
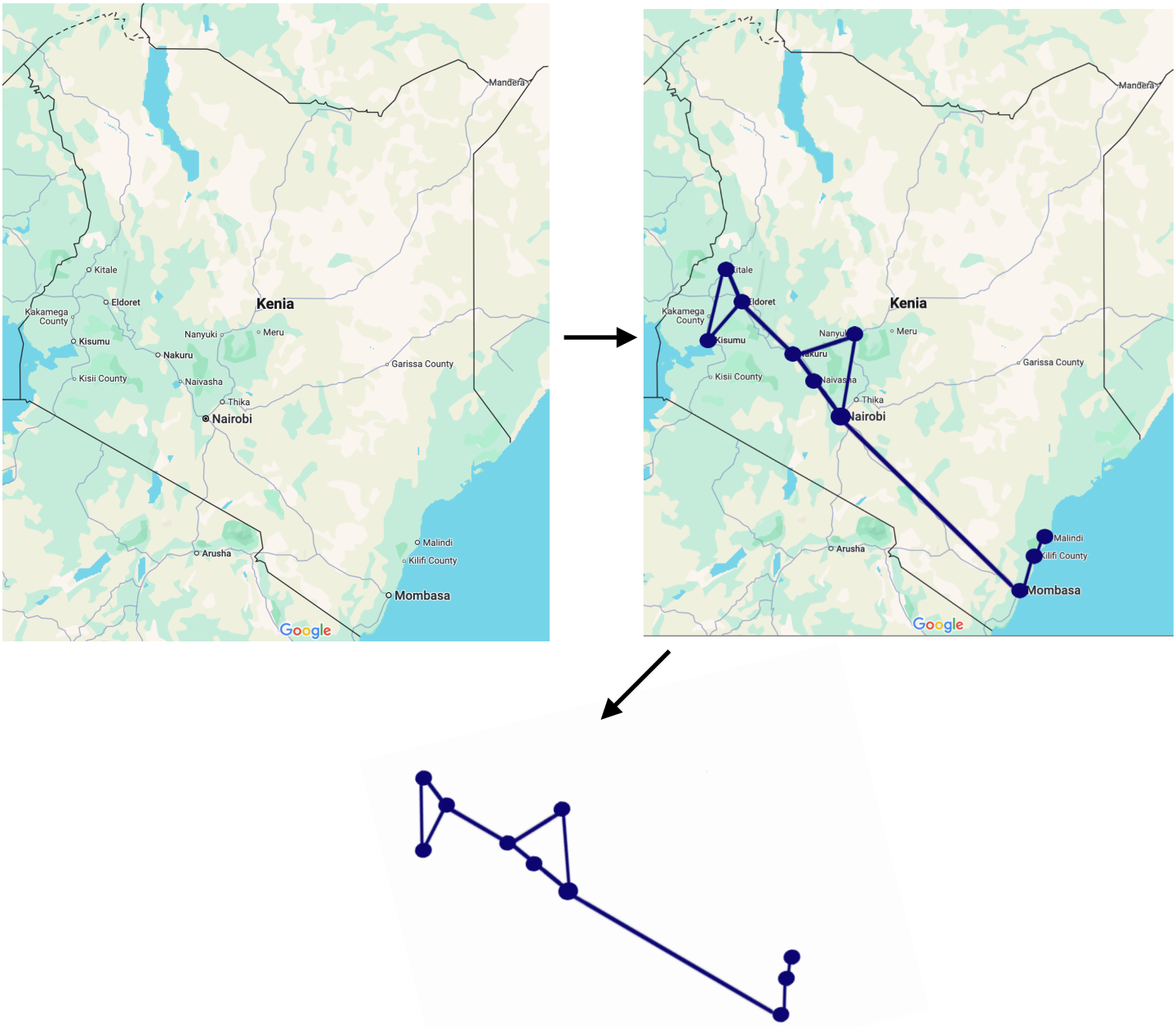


Image: Vienna Metro Map

While the left image looks nicer because it also shows the important places of Vienna, the right one is easier to read. If you want to find the fastest way to get from Karslplatz to Schottenring, for example, this is a lot faster done on the right one. That is why we use graphs - they are very simple and don't show unnecessary information that would distract our attention.

For example, if you wanted to find your way to get around Kenya, you might want to turn a simple map of Kenya into such a graph. All you have to do is make **nodes** for every big city and **edges** for the roads connecting them:



In the first step, you see how the nodes are put on every big city and the edges represent the roads. In the second step, the graph is taken out of the map. Now, if you add the city names, you have a much easier time to find your ways to go from one place to another.

Chapter 3: The Shortest Path

One of the places where we use Graph Theory a lot is in finding shortest paths. This could be on your phone, when you want to find the quickest way to get from A to B. In New York, for example, you might want to know the fastest way to drive from the Empire State Building to Columbia University.

The map would look like this:

Image: Google Maps

How could we get from the Empire State Building in the south to Columbia University in the north? After looking at the map for a while, you might come up with these ideas:

1) You drive straight north past the Central Park and then west to the University.
2) First you drive straight west and then finally straight north to get to Columbia University
3) You drive a bit further west, onto the highway that is in grey on the border of the island and quickly drive slightly north of the University and then make your way to it. This would be longer in distance, but the highway is always faster!

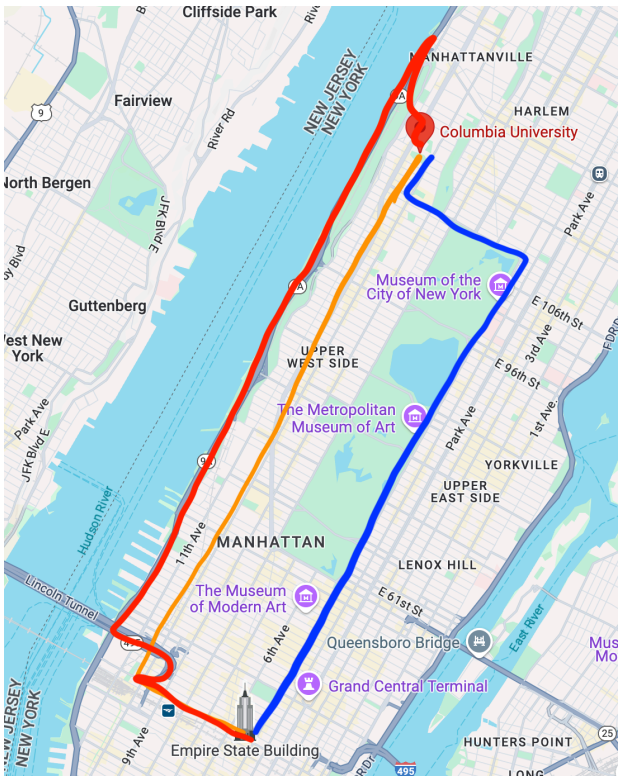


Image: Google Maps

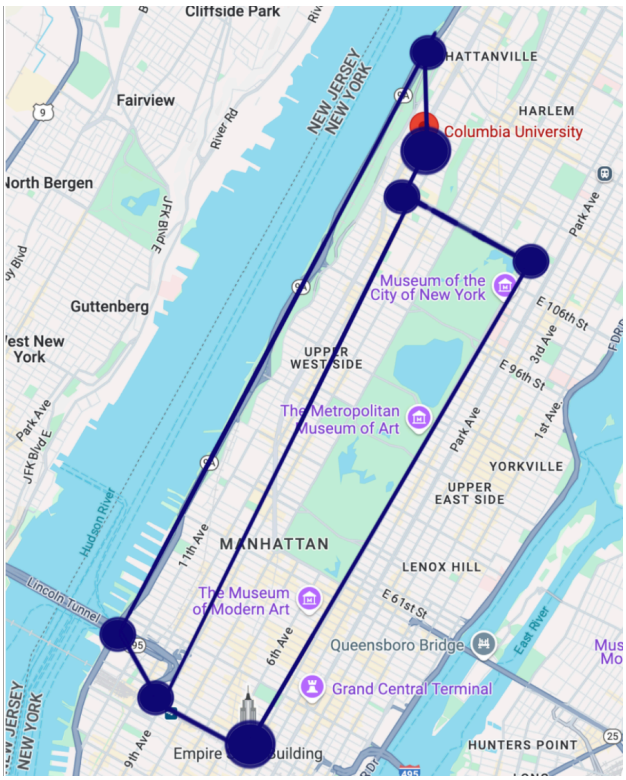
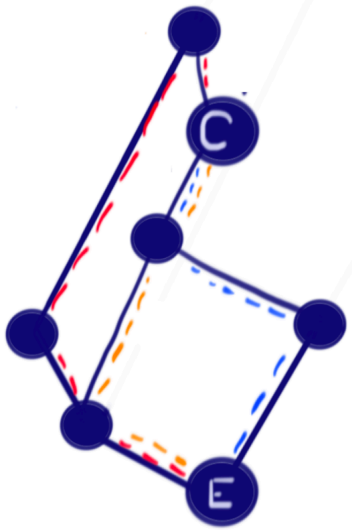
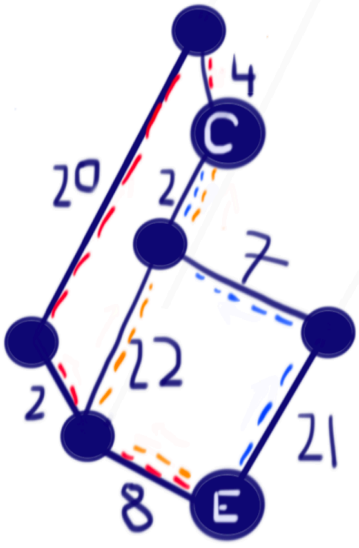


Image: Google Maps

Next, we take the graph from the top right image and make it user-friendly. The lines going north are quite long, so we can shorten them without disturbing the main structure of the graph (see on the left, below). The final step before calculating the shortest path is to add the time it takes to drive each road (on the right, below).





Now we can calculate how long it takes to do each of the three paths:

Path 1: 21+7+2=**30**
Path 2: 8+22+2 = 32
Path 3: 8+2+20+4 = 34

This means the shortest path from the Empire State Building to Columbia University is Path 1.

Exercises Chapter 3

Find the shortest path from A to B (left to right) for each of the following graphs: